CHARACTERIZATION OF FLUCTUATIONS OF CHAOTIC QUANTUM SPECTRA

Oriol Bohigas, Marie-Joya Giannoni and Charles Schmit

Division de Physique Théorique*, Institut de Physique Nucléaire, 91406 Orsay Cedex, France

INTRODUCTION

The problem of finding criteria to characterize quantum chaos has become of interest in recent years, once the essential features of classical chaotic motion have been rather well settled. As will probably appear in this Conference, the tentative definitions of quantum chaos are still sparse and conjectural, reflecting our poor understanding of the question. Our purpose here is to propose signatures of quantum chaos by means of statistical properties of the energy levels of the discrete spectrum. In what follows, we shall consider dynamical systems which are conservative and time-reversal invariant, and their classical analogue strongly chaotic (K-systems).

From different numerical experiments and theoretical studies of such systems performed so far, the main feature that seems to emerge is the presence of the phenomenon of repulsion of levels, i.e. the vanishing of the distribution of the nearest-level spacings $p(x)$ as x goes to zero. This property is well known to occur in the spectra of resonances of the compound nucleus and was first mentioned by Berry and Tabor[1] as being possibly a general feature of levels of irregular spectra, in contrast with the clustering of levels (Poisson distribution $p(x)=e^{-x}$) found by these authors for integrable systems. The first numerical indication (see Fig.1a) of level repulsion for chaotic systems with few degrees of freedom was obtained by McDonald and Kaufman[2] (and confirmed by Casati, Valz-Gris and Guarneri[3]) for the stadium. A calculation by Berry[4] for the Sinai's billiard led to the same conclusion (Fig.1b). Concerning

*Laboratoire Associé au CNRS.

G. Casati (ed.), *Chaotic Behavior in Quantum Systems*

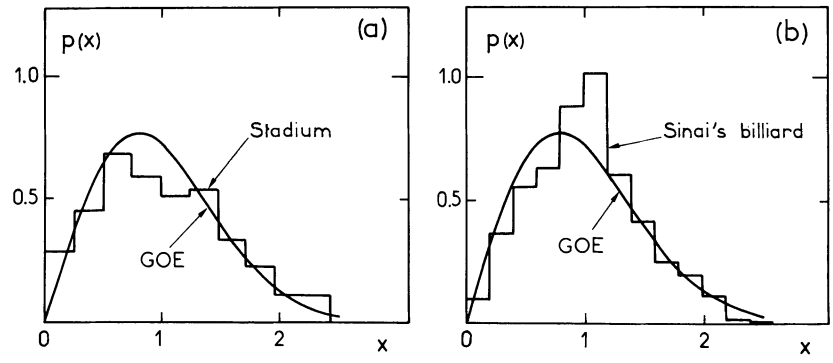

Fig. 1. Distribution of the nearest-neighbour spacings for : a) desymmetrized stadium (histogram taken from Ref.2) ; b) desymmetrized Sinai's billiard (histogram taken from Ref.4). The curve corresponds to the prediction of the Gaussian Orthogonal Ensemble of random matrices, as explained in the text.

theoretical predictions, Zaslavsky guessed[5] that for classical chaotic systems the behaviour of p(x) for small values of x is x^{γ}, where γ depends on the rate of exponential separation of trajectories. However, this conjecture is contradicted by Berry, whose arguments lead to a linear vanishing of p(x) as $x \to o$ for a class of systems ("generic systems") which also include non chaotic systems (see refs. 6,7 and these proceedings). Therefore, it seems that the knowledge of p(x) as $x \to o$ does not provide an unambiguous identification of quantum chaos. On the other hand, it should be noticed that the information carried by p(x=o), although interesting, is quite limited. Many other quantities which are useful in studying level fluctuations have been extensively investigated in the context of Random Matrix Theory (RMT), as for instance, the degree of rigidity of the spectrum, which, in contrast with the level repulsion, does contain information on the correlations between spacings. In Section 2 we give a brief review of the ideas involved in the study of fluctuation properties of quantal spectra[8-9], and of the success of RMT when applied to real physical systems[10,11]. The material presented in Section 2 will be used in Section 3 to study the spectral properties of a classically chaotic system.

2. FLUCTUATION PROPERTIES OF SPECTRA AND RANDOM MATRIX THEORY

a) Parameters Characterizing Fluctuations

In the statistical study of a quantum discrete spectrum, one has to distinguish two kinds of properties : i) global properties, such as the smoothed level density $\bar{\rho}(E)$, and ii) local properties,

i.e. fluctuations of levels around $\bar{\rho}(E)$. These two types of properties are very different in nature and completely disconnected. To study the chaotic behaviour of spectra the average density of states is uninteresting and in what follows we shall concentrate on measures of fluctuations. We first recall that, to get rid of spurious effects on the local properties due to variations of the density, one has to work at constant density on the average. For this purpose it is essential to have a good method providing the smoothed cumulative density $\bar{N}(E)$

$$\bar{N}(E) = \int_0^E \bar{\rho}(E')dE' \tag{1}$$

One can then "unfold" the original spectrum, i.e. map the spectrum of eigenvalues $\{E_i\}$ onto the spectrum $\{\varepsilon_i\}$ through

$$\varepsilon_i = \bar{N}(E_i) \tag{2}$$

We will take as energy unit the average spacing $\bar{x}$ between two adjacent levels of the unfolded spectrum

$$\bar{x} = \bar{s}_i = \overline{(\varepsilon_{i+1} - \varepsilon_i)} \tag{3}$$

The spacing distribution p(x) satisfies then

$$\int p(x)dx = \int xp(x)dx = 1 \tag{4}$$

It should be emphasized that, when studying local properties, we are only interested in results which are translational invariant over the spectrum, i.e. the results in the interval $[\varepsilon,\varepsilon+L]$ should be independent of ε. This may be hard to check when performing numerical studies, especially for systems which reach their asymptotic fluctuation properties at very high excitation energies.

Let us now come to the characterization of fluctuation properties. The first interesting (and most popular) quantity is the spacing distribution p(x) already mentioned and which carries information on the correlation between two adjacent levels. But p(x) tells nothing about the correlations between two adjacent spacings s_i and s_{i+1}, as can be easily realized by constructing a spectrum "level by level" with the prescription that ε_{i+1} has the probability p(x)dx of lying in an interval dx at a distance x of ε_i ; in this way, one gets a spectrum with no correlations between spacings (by taking $p(x)=e^{-x}$ one obtains the Poisson case). The most obvious parameter that can be introduced is the correlation factor C between s_i and s_{i+1}. More interesting is the statistic Δ_3 of Dyson and Mehta[12] :

$$\Delta_3(L;x) = \frac{1}{L} \operatorname*{Min}_{A,B} \int_x^{x+L} [n(\varepsilon) - A\varepsilon - B]^2 \, d\varepsilon , \tag{5}$$

which measures the least-square deviation of the staircase representing the cumulative density $n(\varepsilon)$ from the best straight line fitting it in any interval $[x,x+L]$. If the spectrum is translational invariant, averages of Δ_3 will be independent of the position x of the interval $[x,x+L]$. Δ_3 describes the so-called "degree of rigidity" of the spectrum : the most perfectly rigid spectrum is the picket fence with all spacings equal (for instance, the one-dimensional harmonic oscillator spectrum), therefore maximally correlated, for which $\Delta_3(L) = 1/12$, whereas, at the opposite, the Poisson spectrum ($p(x) = e^{-x}$ and no correlations between spacings) has a very large average value of Δ_3 ($\overline{\Delta}_3(L) = L/15$), reflecting strong fluctuations around the mean level density. Notice that by studying, for instance, the average value of Δ_3 as a function of L, one can choose the range L (in units of the mean spacing) over which fluctuations are investigated. A detailed insight on the information contained in Δ_3 can be found in Refs. 8-10 ; for other fluctuation measures, see e.g. the contribution of S.S.M. Wong to this Conference.

b) Random Matrix Theory : the Gaussian Orthogonal Ensemble (GOE) and its Applications

The idea of representing the Hamiltonian of a complex quantum system by a random matrix was initiated by Wigner, and developed by several authors [8,9] ; the physical purpose of such a theory was to find an appropriate frame to describe the fluctuation properties of slow neutron resonances in heavy nuclei. The ensemble of random matrices which has been the most extensively used in the study of nuclear (and atomic) spectra is the Gaussian Orthogonal Ensemble (GOE). The GOE of NXN real symmetric matrices can be defined in several equivalent ways. Let us remind the main assumptions made in its derivation : the system is time-reversal and rotational invariant (physically justified assumptions), and all matrix elements H_{ij} ($i \leq j$) are independent random variables (for reasons of mathematical tractability but physically unjustified). In the limiting case when $N \to \infty$, it is possible to get analytical results for several properties. In particular, for the quantities considered in 2.a) one has : i) p(x) is very accurately approximated by the so-called Wigner surmise :

$$p(x) \simeq \frac{\pi}{2} x \exp\left(-\frac{\pi}{4} x^2\right) ; \qquad (6)$$

ii) the correlation factor C between adjacent spacings has the average value

$$\overline{C} = -0.271 \qquad (7)$$

iii) the average value of Δ_3 bahaves asymptotically (large L) as :

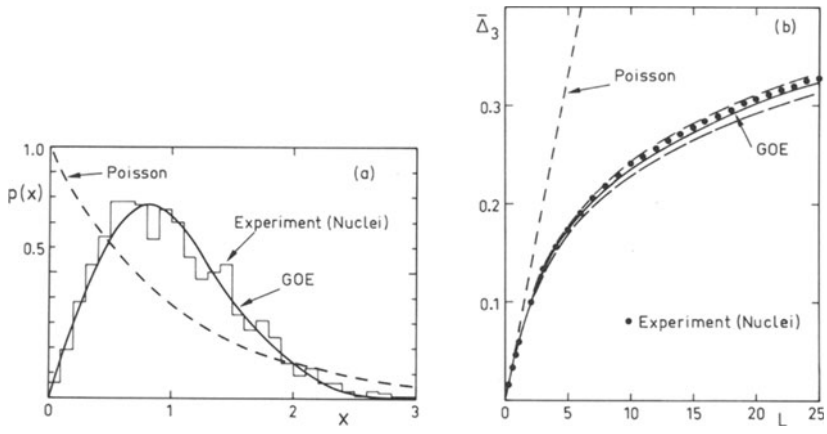

Fig. 2. Fluctuation properties of nuclear levels (taken from Ref. 10). 1762 resonance energies have been included in the analysis ; a) nearest-neighbour spacing histogram, b) average value of Δ_3 as a function of L. The curves corresponding to the Poisson (uncorrelated) and GOE cases are plotted for comparison. Dashed lines close to the GOE values of $\bar{\Delta}_3$ correspond to one standard deviation when finite sampling effects, provided by GOE, are taken into account.

$$\bar{\Delta}_3(L) \simeq \frac{1}{\pi^2} \ln L - 0.007 \qquad (8)$$

One can obtain the exact value[10] of $\bar{\Delta}_3(L)$ by a numerical integration, using the known expression of the two point correlation function[13]. For L = 5 the value obtained from Eq.(8) differs from the exact one by 10% and for L = 15 by 2.5%. The most salient feature of GOE fluctuation properties is a very strong spectral rigidity, which shows up in the logarithmic behaviour of $\bar{\Delta}_3$ (compare to the linear behaviour for the Poisson case).

What is, now, the predicting power of GOE when applied to level fluctuations of complex nuclei? Recently, the whole body of accurate nuclear data (~ 1750 resonances coming from ~ 30 different nuclei) has been analyzed simultaneously and several new spectral measures have been used[10]. The results for p(x) and $\bar{\Delta}_3$ ($0 < L < 25$) are reproduced in Fig. 2. Other fluctuation measures have been considered and the comparison between GOE predictions and experimental data has yielded a remarkable agreement, even for very sensitive quantities like the variance of Δ_3. These results provide a conclusive evidence for the validity of GOE fluctuations, raising again the old question : why is this parameter-free theory, in which no information about the specific features of the system is included (as, for instance, the interaction between nucleons), so efficient? Does the success of GOE denote that fluctuation properties of nuclear levels

Table 1. Fluctuations of Atomic Energy Levels.

	Nd	Nd	Nd^+	Nd^+	Nd^+	Sm^+	Sm^+	Tb
J^π	4^-	6^-	$7/2^-$	$13/2^-$	$15/2^-$	$3/2^-$	$9/2^-$	$9/2^-$
L	35	38	34	28	32	26	31	45
$\overline{\Delta}_3(L)$ exp.	0.39	0.45	0.30	0.37	0.39	0.37	0.40	0.31
$\overline{\Delta}_3(L)$ GOE	0.35 ±0.11	0.36 ±0.11	0.35 ±0.11	0.33 ±0.11	0.34 ±0.11	0.32 ±0.11	0.34 ±0.11	0.38 ±0.11

Results obtained by Camarda and Georgopulos[11] for Δ_3 for different series of levels of neutral and singly ionized atoms. Each series is identified by the angular momentum and parity (J^π). For GOE the value of $\overline{\Delta}_3$ is followed by the square root of the ensemble average of the variance of Δ_3 (asymptotic value =0.11).

result from a general law of nature? That this is probably the case is corroborated by a recent analysis[11] of 269 atomic energy levels corresponding to 8 different atoms. We reproduce in Table 1 some of the results for Δ_3. One can see that the agreement between theory and experiment is good.

3. LEVEL FLUCTUATIONS OF QUANTUM SINAI'S BILLIARD

We present now numerical results for the level fluctuations of the desymmetrized quantum Sinai's billiard (see Fig. 3a). Use will be made of the methods outlined before. We proceed as follows : We determine the eigenvalues $E_i = k_i^2/2m$ of the Schrödinger equation

$$(\Delta + k_i^2)\,\psi_i = 0 \qquad (9)$$

with Dirichlet boundary conditions by using the method of Korringa-Kohn-Rostoker as described in Ref.4. We compute several sets of eigenvalues $\{E_i(R)\}$ for different values of R. To unfold the spectrum we use the Weyl-type formula which gives the average number of levels up to energy E[14]

$$\overline{N}(E) = \frac{1}{4\pi}\,(SE - L\sqrt{E} + K)\,, \qquad (10)$$

where S and L are respectively the surface and the perimeter of the billiard and K is a constant of the order of unity. To insure

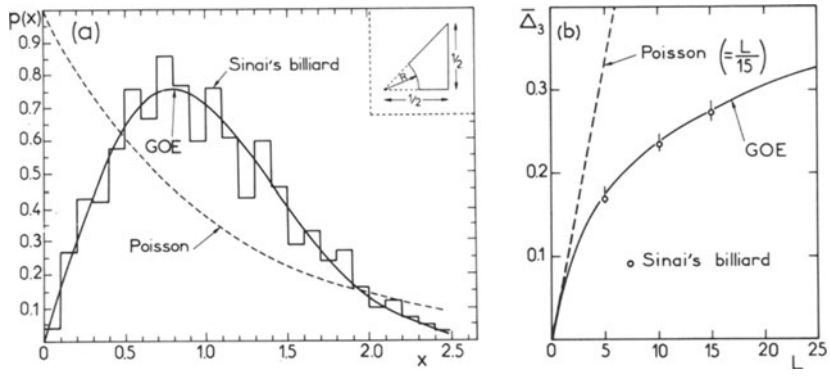

Fig. 3. Results of energy level fluctuations for desymmetrized Sinai's billiards as specified on the upper right corner of Fig. 3a. 740 levels have been included in the analysis corresponding to the 51-th to 268-th level for R = 0.1, 21-th to 241-th level for R = 0.2, 16-th to 194-th level for R = 0.3, 11-th to 132-th level for R = 0.4. Fig. 3a : results for the spacing distribution p(x). Fig. 3b : results for the average value of the $\Delta_3(L)$ statistic of Dyson and Mehta for L = 5, 10 and 15. Curves corresponding to the Poisson case (stretch of uncorrelated levels) and to the random matrix theory predictions (GOE) are drawn for comparison. The error bars on Fig. 3b (one standard deviation) correspond to finite sampling effects as predicted by GOE.

translational invariance we have excluded from the final analysis the first levels of every spectrum. This is natural if one observes the spectra corresponding to different values of R : one realizes that the smaller the value of R the more the number of levels, starting from the ground state, that can be obtained by perturbing the spectrum $\{\varepsilon_i(R=0)\}$ (triangular billiard). And, as it will come out from our analysis, the fluctuation properties of Sinai's billiards are essentially different from the triangular billiard and are characteristic of levels that cannot be attained by perturbation, i.e., of levels that are at relatively high energy. To increase the statistical significance of the results several spectra $\{\varepsilon_i(R)\}$ corresponding to four different values of R have been analyzed as corresponding to a single stretch of 740 levels. In doing so, care has been taken that one is working with "independent information" : the different values of R should not be chosen to be too close to one another. Otherwise two different spectra corresponding to R and $R+\delta R$ would be almost deducible one from the other and one would just be dealing with redundant information.

In Fig. 3a is shown the spacing distribution p(x) which compares beautifully with the GOE prediction not only for small spacings (level repulsion) but over the entire range of spacings. As can be seen in Fig. 1b, the results obtained in Ref.4 do show departures

Table 2. Fluctuation measures of Sinai's billiard

	Sinai	GOE	Poisson
C	-0.30	-0.27 ± 0.04	0.0
σ^2	0.273	0.286 ± 0.015	1.0

Correlation factor C between adjacent spacings and variance σ^2 of the spacing distribution p(x). See caption of Fig. 3 for further explanation.

from GOE predictions : this is due to lack of translational invariance in the spectra considered (only the lowest ~ 20 first levels of each spectrum were included) and probably also to redundancy in the spectra (too close values of R were considered). In Table 2 the variance σ^2 of p(x) is reproduced for Sinai's billiards and shown to be consistent with the GOE prediction. We consider next spacing correlations. In Fig. 3b are shown the results for the average value of $\Delta_3(L)$ for $L = 5$, 10 and 15 which again compare remarkably well with the corresponding GOE predictions. In Table 2 is shown the correlation factor between two neighbour spacings which again agrees with GOE. We therefore conclude that all level fluctuation measures of Sinai's billiard investigated so far are consistent with GOE predictions.

4. SUMMARY AND CONCLUSIONS

The present work gives a short review of the remarkable success of the GOE in predicting nuclear and atomic energy level fluctuations. Some of the more characteristic properties of GOE fluctuations are described. It is emphasized that the repulsion of levels does not exhaust, by any means, the richness of GOE fluctuations. Indeed, properties like the rigidity of the spectrum are of most relevance. In contrast, in the previous studies of spectral behaviour of regular and irregular systems, attention has been exclusively paid to the presence or absence of level repulsion (which is unable to distinguish between chaotic and non chaotic systems). We propose to use the tools developed in random matrix theory for searching signatures of chaotic spectra.

We then proceed to study numerically the level fluctuations of Sinai's quantum billiard (a chaotic system). We find that all fluctuation measures investigated are consistent with the corresponding GOE predictions. One can then conjecture that chaotic systems (which

ones should be made precise in the future) show GOE fluctuations. The correctness of the conjecture would bring a new perspective in the understanding of the origin and domain of validity of GOE fluctuations.

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