Vectorial Space Structure of the Set of Cycles of a Graph

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Abstract

We investigate the structure of the cycle space of a graph and show that it is possible to assign to it a vector space over $GF(2)$, with the symmetric difference as the composition rule.

1 Introduction

In 1935 Schrödinger defined entanglement as the distinctive property of quantum mechanics [1]. It is interesting to note that, in addition to the usual definition in terms of the separability of a composite system state (the "representative" in his words) into a product of individual factors corresponding to each constituent, he put forward the fact that the information contained in the whole cannot be necessarily obtained from the information contained in its parts, implicitly expressing that entanglement is an information resource. Since then the double status of the quantum state as physical and informational has gained in importance to characterize a great variety of phenomena [2, 3], ranging from quantum computing [4, 5] to topological phases in condensed matter [6, 7].

2 Model

A cycle b is a closed path in G; it is a subset of the graph edge set E. The set $B_C(G)$ of all cycles is the cycle space. To each cycle $b \in B_C(G)$ we can associate a vector with $|E|$ components, each taking the values in the set $\{0, 1\}$, where the value 1 stands for an edge in b, and 0 otherwise. The cycle space $B_C(G)$ equipped with the ring sum forms a vector space over the finite field GF(2), of dimension [8],

$$|B| = |E| - |V| + 1 \tag{1}$$

(for a connected graph). The composition rule of the vector space, the ring sum $b_1 \oplus b_2$, corresponds to the symmetric difference between the edges subsets $(b_1 \cup b_2)$ $(b_1 \cap b_2)$. A set of cycles that cannot be written one another as a ring sum combination, is an independent set. A set $B \subset B_C$ of independent cycles of dimension $|B|$, given by (1), is a cycle basis B:

$$B = \{b_n \in B_C,\ n = 1, \dots, |B|\}\,, \tag{2}$$

of B_C. Therefore, every element $b \in B_C$ can be written as a linear combination of the basis cycles:

$$b = \sum_{n=1}^{|B|} s_n b_n, \quad s_n = 0, 1\,.$$

We call the minimum cycle basis of B_C, the basis of shortest length [9],

$$B_G = \{b_n \in B \,|\, \min \mathrm{len}(B)\} \tag{3}$$

where,

$$\mathrm{len}(B) = \sum_{n=1}^{|B|} \mathrm{len}(b_n)$$

with the length of a cycle b defined as the total number of its nodes:

$$\mathrm{len}(b) = |b|\,.$$

3 Conclusions

Using a model of an interacting quantum system introduced in [10], we investigated the entanglement structure of the thermal state. The model do not contain dimensional parameters: it is essentially defined by the graph...

References

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