THEORY OF A WEAKLY TURBULENT PLASMA

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§ 1. Plasmon – Particle Interactions

A characteristic feature of a plasma is the existence of a spectrum of collective oscillations or plasma waves (plasmons). The frequency and velocity of propagation of these waves are determined by the wave vector and by the gross parameters of the plasma such as the density, the mean velocity spread, the magnetic field, etc., and this situation is a reflection of the fact that all of the particles in the plasma are involved in the plasma oscillations. The situation is different, however, when one examines the damping (or growth) of the oscillations. Damping (growth) is determined by the "fine details" of the particle distribution in phase space, for example, by the derivative of the velocity distribution function; this situation reflects the specific role played by resonance particles (i.e., particles for which the following condition is satisfied: $\omega_{\mathbf{k}} - \mathbf{kv} = n\omega_H$; $n = 0, 1, 2, \ldots$; here, $\omega_{\mathbf{k}}$ and $\mathbf{k}$ are the frequency and wave vector that characterize the wave, $\mathbf{v}$ is the particle velocity, and $\omega_H = eH/mc$). These particles are capable of exchanging energy with the waves and can thus amplify or damp it.

The important role played by resonance particles in the damping of plasma waves is evident from the fact that the damping of a wave characterized by frequency ω and wave number k in a plasma in thermodynamic equilibrium is found to be proportional to the derivative of the electron distribution function $f'(v)$ taken at the point $v = \omega_k/k$; this result was first established by Landau [1] through the use of the self-consistent field equations [2,3].

Many later authors have verified this result, i.e., that the damping rate (or growth rate) for waves in a low-density plasma is proportional to the derivative of the distribution function for the resonance particles (cf. [4]).

A detailed physical analysis of the interaction of the plasma particles with plasma waves (as well as problems touching on the propagation of plasma waves) has been given by Bohm, Gross, and Pines [5], who have indicated the importance of resonance particles for a given wave mode. The importance of resonance particles in both absorption and growth of plasma waves may be regarded as well established.

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We shall also be concerned with the effect of plasma waves on transport processes in a low-density plasma. This question arises in connection with the problem of treating the "Coulomb logarithm" in the collision term that appears in the kinetic equation for a low-density plasma [3, 6]. Davydov [7] has estimated the contribution of plasma waves to the kinetic coefficients for a plasma close to a state of thermodynamic equilibrium and has shown that taking account of the emission and absorption of plasma waves by particles (in addition to the usual binary collisions between particles) can modify the value of the Coulomb logarithm substantially.*

It is clear, however, that treating these two processes separately in a plasma close to thermodynamic equilibrium is not consistent with the available accuracy, since the exact values of the quantities that appear in the Coulomb logarithm remain unknown when this approach is used. In order to make a more consistent formal calculation [12], the kinetic coefficients associated with collisions between particles and with the emission and absorption of waves by particles cannot be treated separately – they must be treated together.

The situation is different, however, if one considers a weakly turbulent plasma in which the energy density contained in the waves (plasma oscillations) is small compared with the thermal energy density, but appreciably greater than the energy density associated with the equilibrium thermodynamic plasma noise (this situation is frequently realized in low-density plasmas, cf. below). Under these conditions, one need not necessarily take account of collisions between particles at the outset; the plasma can be treated by means of the self-consistent field equations. It turns out that these equations can be replaced by the simpler equations of the quasi-linear theory [4, 13-15]: an equation for the rate of energy growth (damping) of the plasma waves, and a diffusion-like equation for the distribution function of the resonance particles in the plasma (the diffusion coefficient being proportional to the energy density of the waves in the turbulent plasma).

It must be emphasized that this derivation of the quasi-linear equations from the self-consistent field equations can only be carried through when the

*The ideas in [7] have been developed further by Galitskii and also by Romanov and Filippov [9], who have derived a system of kinetic equations for a system of electrons and plasma waves by analogy with the kinetic equations for electrons and phonons in a solid. Similar equations have been studied by Silin and Klimontovich [10] and by Pines and Schriffer [11].

resonance particles comprise a small group which does not have an important effect on the "gross" characteristics of the plasma (density, temperature, etc.).

The quasi-linear theory, which is described briefly below, describes the dynamics of the interaction between the resonance particles and the waves. A consistent derivation of the equations and an analysis of these processes can be carried out only when the energy contained in the collective degrees of freedom, i.e., the plasma oscillations, is much smaller than the energy associated with the random motion of all the particles (but, at the same time, greater than the energy associated with the thermal noise in the collective degrees of freedom).

The essence of the quasi-linear method lies in separating the distribution function for the resonance particles into a rapidly varying part and a slowly varying part, and then taking account of the average quadratic effect of the rapidly varying part on the slowly varying part (the method is similar to the well-known method of Van der Pol used in nonlinear mechanics). When this is done, it turns out that the behavior of the slow part of the distribution function is described by a diffusion equation in phase space and that the rate of growth (or damping) of the fast oscillations (plasma waves) is determined by the formulas of the linear theory, with the nonoscillating part of the distribution function varying slowly in time.

In a homogeneous low-density plasma in which collisions between particles are not important, there is a large degree of arbitrariness in the choice of the stationary velocity distribution function. The quasi-linear theory indicates the existence of well-defined states to which the unstable plasma evolves as a result of the development of perturbations.

These states are characterized by the fact that in certain regions of phase space the distribution function f becomes a constant (i.e., a plateau appears on the function f); in the corresponding regions of wave-number space the plasma oscillations are present in the form of noise at a level appreciably greater than the thermal level.

§ 2. Basic Equations of the Quasi-Linear Theory

We shall first derive the basic equations for the quasi-linear theory assuming a fully ionized, low-density plasma. We assume that the distribution function f_α for particles of species α with charge e_α and mass m_α obeys the self-consistent field equations

$$\frac{\partial f_\alpha}{\partial t} + v\frac{\partial f_\alpha}{\partial x} + \frac{e_\alpha}{m_\alpha}\left(E + \frac{v}{c} \times H\right)\frac{\partial f_\alpha}{\partial v} = 0, \tag{1}$$

where the self-consistent fields E and H are determined by the distribution of plasma particles:

$$\left.\begin{aligned} \frac{\partial E}{\partial t} &= 4\pi \sum_{\alpha} e_{\alpha} \int f_{\alpha}\, dv; \quad \nabla \times E = -c^{-1}\, \partial H/\partial t; \\ \nabla \times H &= 4\pi c^{-1} \sum_{\alpha} e_{\alpha} \int v f_{\alpha}\, dv + c^{-1}\, \partial E/\partial t. \end{aligned}\right\} \tag{2}$$

The system of self-consistent field equations (1)-(2) yields a proper description of the plasma if the plasma is almost ideal,* i.e., if the average amplitude of the Coulomb scattering e^2/T (where T is the mean kinetic energy of a particle) is much smaller than the mean distance between particles $r \approx n^{-1/3}$ (where n is the plasma density). Under these conditions, the number of particles in a Debye sphere $N_D \approx n(T/ne^2)^{3/2}$ is appreciably greater than unity and the quantity N_D^{-1} is a small parameter in terms of which one usually expands the exact equations of motion for the plasma particles; in the first approximation this procedure leads to the system (1)-(2) and in higher approximations to the appearance of a collision term on the right side of Eq. (1) [3]. However, if one is interested in the dynamics of such a plasma,† it turns out that the parameter N_D^{-1} is not the only small parameter in a low-density plasma. A low-density plasma can exhibit plasma oscillations of various kinds. In the absence of a magnetic field, these are the electron plasma oscillations and the ion-acoustic oscillations; the frequencies and propagation velocities of these waves are determined by the gross properties of the plasma (the density, mean velocity spread, etc.). The damping (or growth rate) for these waves depends on the fine details of the distribution function in phase space. The plasma particles experience the random effect of the electric fields associated with many waves and diffuse in phase space, and under these conditions changes occur in precisely those details of the distribution function which are responsible for the wave damping. On the other hand, the gross properties of this system are not changed in these processes; the wave energy

*In addition we assume that quantum degeneracy effects are not important (this condition imposes a further limitation on the plasma density: degeneracy effects can be neglected if $\lambda \ll r$, where $\lambda \approx \hbar/mv$ is the mean wavelength associated with the particle).

†This is in contrast with the case of an ideal plasma in thermodynamic equilibrium in which the ratio of the mean scattering amplitude for binary collisions to the mean distance between particles is the only small parameter, and in terms of which a consistent expansion procedure can be used to find the equation of state [17].

is so small that there is no appreciable effect on the mean plasma density, on the various moments of the distribution function, etc.

The particle diffusion rate in velocity space as a result of the waves is proportional to the energy density in the waves ε. If the ratio ε/nT is appreciably greater than N_D^{-1}, the effects due to "wave" diffusion are greater than those due to collisions between particles (which also cause diffusion in velocity space); in this case collisions between particles can be neglected in a first approximation.

The ratio of the energy density in the nonequilibrium plasma oscillations to the kinetic energy density ε/nT represents the second small parameter in the dynamics of a low-density plasma. In what follows we assume that

$$1 \gg \frac{\varepsilon}{nT} \gg N_D^{-1},$$

implying that the energy density in the collective degrees of freedom of the plasma-wave exceeds appreciably the density in the Coulomb interactions: $\varepsilon \gg nT/N_D$.* At the same time, ε is much smaller than the thermal energy nT.

The equations of the quasi-linear theory are obtained by expanding the self-consistent field equations (1)-(2) in terms of the small parameter ε/nT, and taking account of terms that are quadratic in the amplitude of the plasma oscillations.

To be definite, let us consider the case of longitudinal plasma oscillations in a plasma in the absence of a magnetic field. The point of departure is the self-consistent field equation for the electron distribution function

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{eE}{m}\frac{\partial f}{\partial v} \tag{3}$$

and the equation

$$4\pi en \int vf\,dv = -\partial E/\partial t\,. \tag{4}$$

(Here, n is the equilibrium plasma density, so that $\int f_0 dv = 1$ at equilibrium.) As is well known from the linear theory, the harmonic (in time) solutions of (3) and (4) describe plasma oscillations; the damping rate for these oscillations in an equilibrium plasma is small compared to the frequency if the

*To within an accuracy of order unity the quantity nT/N_D is equal to the familiar Debye correction to the free energy of a Coulomb particle system; the same quantity characterizes the energy density of the thermal plasma waves.

wavelength is much greater than the Debye length. We can write the linearized equations (3)-(4) in terms of Fourier space components

$$f = f_0 + \sum_k G_k e^{ikx};$$

$$E = \sum_k E_k e^{ikx},$$

and find [neglecting the $v(\partial f/\partial x)$ term at long wavelengths]

$$\dot{G}_k = -\frac{e}{m} E_k \frac{\partial f_0}{\partial v};$$

$$\dot{E}_k = -4\pi e n \int v G_k \, dv,$$

whence it follows that the longwave component of the field satisfies an oscillator equation

$$\ddot{E}_k = -\omega_p^2 E_k \quad (5)$$

characterized by the plasma frequency $\omega_p = (4\pi n e^2/m)^{\frac{1}{2}}$. All of the plasma particles participate in these oscillations and the kinetic energy associated with the motion of all the particles in this wave is equal to the electrostatic energy (virial theorem):

$$\frac{1}{2} nm \left| \int v G_k \, dv \right|^2 = \frac{E_k^2}{8\pi},$$

so that the total energy associated with the oscillations is $E_k^2/4\pi$. Oscillations for which the wave number k is large are not damped if there are no particles characterized by velocities $v \gg \omega_p/k$ [under these conditions, we can neglect the term $v\partial f/\partial x \sim kvG_k$ as compared with $\partial f/\partial t \sim \omega_p G_k$, as has been done in deriving Eq. (5)].

On the other hand, if the plasma does contain electrons whose velocity coincides with the phase velocity of any of the plasma waves ω_p/k, it then becomes possible to have an energy exchange between the waves and these "resonance" particles. We shall assume that the number of resonance particles is small and neglect any effect they may have on the dispersion properties of the plasma (the oscillation frequency, the phase velocity, and the group velocity – but not the damping!). In the case being considered here, this means that in the presence of the resonance electrons we still take the frequency of the plasma oscillations to be ω_p.

The interaction between the plasma oscillations and the resonance particles leads to two effects: first, there is a change in the mean energy of the plasma oscillators $E_k^2/4\pi$; second, there is a simultaneous change in the distribution of resonance electrons in velocity space. To derive equations that describe these processes we proceed as follows.

We write the distribution function for the resonance particles F in the form of a sum of a rapidly oscillating term $\sum_k F_k e^{ikx}$ and a slowly varying function $\bar{F}$; in this case, the electric field $E = \sum_k E_k e^{ikx}$ is in the form of a product of rapidly oscillating space-time functions multiplied into a slowly varying amplitude (we assume the mean field to be zero). The average taken over a time period much greater than the period of the plasma oscillations leads to zero values for the oscillating parts of the distribution function and the electric field:

$$\langle F_k \rangle = \langle E \rangle = 0,$$

so that $\bar{F}$ represents the mean value of the total distribution function for the resonance electrons F. In the kinetic equation for the resonance electrons,

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} = -\frac{eE}{m}\frac{\partial F}{\partial v} \tag{6}$$

we now take an average over space:

$$\frac{\partial \bar{F}}{\partial t} = -\overline{\frac{eE}{m}\frac{\partial F}{\partial v}} = -\frac{\partial}{\partial v}\frac{e}{m}\sum_k E_k^+ F_k. \tag{7a}$$

Subtracting Eq. (7a) from Eq. (6), and neglecting the difference* $E\partial F/\partial v - \overline{E\partial F/\partial v}$, we find

$$\dot{F}_k + ikvF_k = -\frac{eE_k}{m}\frac{\partial \bar{F}}{\partial v}. \tag{7b}$$

From the kinetic equations for the nonresonance particles, we have

$$\dot{G}_k = -\frac{eE_k}{m}\frac{\partial f_0}{\partial v},$$

*In Eq. (7b) this difference would yield terms that are nonlinear in E, these terms representing the interaction of plasma waves with themselves. This effect can be neglected for weak excitation of the plasma.

so that the time derivative of the current density produced by all the plasma particles (except for the resonance particles) is

$$ne\int vG_k = \frac{ne^2}{m}E_k.$$

Making use of this relation and using the equation for the total current (4), we have

$$\ddot{E}_k + \omega_p^2 E_k = 4\pi en\int v\dot{F}_k\,dv. \tag{7c}$$

We now integrate Eq. (7b)

$$F_k(t) = F_k(0)\,\mathrm{e}^{-ikvt} - \frac{e}{m}\int_0^t dt' E_k(t')\,\frac{\partial \overline{F}(t')}{\partial v}\,\mathrm{e}^{ikv(t-t')}$$

and substitute this expression in Eq. (7c); multiplying both sides of the resulting relation by $\dot{E}_k^+$ and adding the complex conjugate, we have

$$\dot{E}_k^+(\ddot{E}_k + \omega_p^2 E_K) + c.\,c. = \frac{d}{dt}\left(|\dot{E}_k|^2 + \omega_p^2\,|E_k^2|\right) =$$

$$= 4\pi en\dot{E}_k^+\int v\,dv\left\{-ikvF_k(0)\,\mathrm{e}^{-ikvt} - \frac{eE_k(t)}{m}\,\frac{\partial\overline{F}(t)}{\partial v} + \right.$$

$$\left. + \frac{e}{m}\,ikv\int_0^t dt' E_k(t')\,\frac{\partial\overline{F}(t')}{\partial v}\,\mathrm{e}^{-ikv(t-t')}\right\} + c.\,c.$$

Substituting the relation $E_k(t) = \sqrt{\varepsilon_k(t)}e^{-i\omega_p t}$ in this expression, and taking the slowly varying functions ε_k and $\partial\overline{F}/\partial v$ outside the integral over t', we obtain an equation for the square of the wave amplitude:

$$\frac{d}{dt}\,2\omega_p^2\varepsilon_k = -4\pi nei\omega_p\sqrt{\varepsilon_k}\int v\,dv\int\left\{-ikvF_k(0)\,\mathrm{e}^{-ikvt} + \right.$$

$$\left. + ikv\frac{e}{m}\sqrt{\varepsilon_k}\,\frac{\partial\overline{F}}{\partial v}\int_0^t \mathrm{e}^{i(\omega_p - kv)(t-t')}\,dt'\right\} + c.\,c.$$

Now, going to the limit $t \to \infty$, we find that the first term in the curly brackets vanishes and that the second, in accordance with the formula

$$\lim_{t\to\infty}\left(\int_0^t \mathrm{e}^{i\alpha(t-t')}\,dt' + c.\,c.\right) = 2\pi\delta(\alpha),$$

yields

$$\frac{d\varepsilon_k}{dt} = \varepsilon_k \pi \omega_p^2 \int v \frac{\partial \overline{F}}{\partial v} \delta(\omega_p - kv)\, dv,$$

i.e., the growth rate for the energy in a given harmonic

$$\frac{d\varepsilon_k}{dt} = 2\gamma_k \varepsilon_k, \tag{8}$$

where

$$\gamma_k = \frac{\pi}{2} \frac{\omega_p^3}{k^2} \int k \frac{\partial \overline{F}}{\partial v} \delta(\omega_p - kv)\, dv. \tag{8a}$$

Thus, in the quasi-linear theory the rate of growth (or damping) of the energy of a given Fourier harmonic is determined by the formula of the linear theory except that the unperturbed "linear" distribution function in the expression for the growth rate (damping) is replaced by the averaged function $\overline{F}$.

The second equation in the quasi-linear theory is obtained by substituting the following expression in Eq. (7a):

$$F_k(t) = F_k(0)\, e^{-ikvt} - \frac{e}{m} \int_0^t dt' E_k(t') \frac{\partial \overline{F}(t')}{\partial v} e^{-ikv(t-t')}$$

and adding the resulting expression to its complex conjugate:

$$\frac{\partial \overline{F}}{\partial t} = -\frac{1}{2} \frac{\partial}{\partial v} \frac{e}{m} \sum_k E_k^{+} \Big\{ F_k(0)\, e^{-ikvt} - $$

$$- \frac{e}{m} \int_0^t dt' E_k(t') \frac{\partial \overline{F}(t')}{\partial v} e^{-ikv(t-t')} \Big\} + c.c.$$

Replacing $E_k(t)$ by $\sqrt{\varepsilon_k(t)}\, e^{-i\omega_k t}$ as in the above, we now obtain the following equation for the averaged function describing the distribution of resonance particles $\overline{F}$:

$$\frac{\partial \overline{F}}{\partial t} = \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k \varepsilon_k \pi \delta(\omega_p - kv) \frac{\partial \overline{F}}{\partial v}.$$

Thus, the second equation is of the form [4, 14, 15]

$$\frac{\partial \bar{F}}{\partial t} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial \bar{F}}{\partial v_\beta}, \tag{9}$$

where

$$D_{\alpha\beta} = \pi \frac{e^2}{m^2} \sum \varepsilon_k \delta(\omega_k - kv) \frac{k_\alpha k_\beta}{k^2}. \tag{9a}$$

Equations (8) and (9) represent a closed* system of equations of the quasi-linear theory for the spectral density ε_k and the averaged distribution function $\bar{F}(v)$.† Equation (9) is in the form of a diffusion equation in which the diffusion coefficient D is proportional [as follows from Eq. (9a)] to the energy density of the plasma waves which, in turn, depends on the distribution function.

The system of quasi-linear equations (8)-(9) which has been obtained from the self-consistent field equations (3)-(4) obviously contains less information than the original equations (for instance, we can only find the amplitudes $\sqrt{\varepsilon_k}$ and not the phases of the fast oscillations). Furthermore, the region of applicability of the quasi-linear theory is much narrower than that of the original system. However, these shortcomings are balanced by the relative simplicity of the equations of the quasi-linear theory.

The system of quasi-linear equations (8)-(9) which describes the interaction of resonance particles with plasma waves exhibits an energy integral. Let us consider the time derivative of the total energy of the system of resonance particles and waves Q. The quantity Q is made up of the kinetic energy of the resonance electrons, the electrostatic energy in the plasma waves $\sum_k \frac{\varepsilon_k}{8\pi}$ and the kinetic energy of all the plasma electrons that participate in the oscillations; by the virial theorem the latter energy is equal to the electrostatic field energy. Thus,

$$\frac{dQ}{dt} = \frac{d}{dt}\left(n \int \frac{mv^2}{2} F\, dv + \sum_k \frac{\varepsilon_k}{4\pi}\right).$$

*The dependence of ω_k on k is determined by the gross parameters of the plasma and is assumed to be known.

†Hereinafter the averaging symbol will be omitted over the distribution function.

Substituting the value of $\partial F/\partial t$ from Eq. (9) and the value of $d\varepsilon/dt$ from Eq. (8) and integrating by parts, we find

$$\frac{dQ}{dt} = 0,$$

i.e., the total energy of the plasmon—particle system is conserved.

In order to understand the physical meaning of the quasi-linear theory, and in order to generalize the equations that have been obtained [(8)-(9)], we view a plasma with highly excited collective degrees of freedom as a system of two gases: a particle gas (fermions), which we will assume to be nondegenerate, and a plasmon gas (bosons).

We now consider the equation showing the balance in the number of particles and waves in phase space assuming that the system is homogeneous and that the condition $N_D^{-1} \ll \varepsilon/nT \ll 1$ is satisfied (the density of the wave gas is considerably greater than the thermodynamic equilibrium value). Since the particle—particle and wave—wave interactions are unimportant,* to a first approximation we need only consider the interaction between particles and waves.

The basic process which we wish to consider is the first-order radiation (Fig. 1a) or absorption (Fig. 1b) of a plasmon q by a particle k.

The process denoted *a* is the Cerenkov emission of a plasmon by an electron moving in the plasma with a velocity v which exceeds the phase velocity of the plasma wave ω_k/k:

$$v = \frac{\omega_k}{k}\,\frac{1}{\cos\vartheta};$$

the inverse process b is the Cerenkov absorption of a plasmon by a particle.

In the case we are considering, in which the density of waves in phase space N_q is large, the matrix elements for these processes are proportional to

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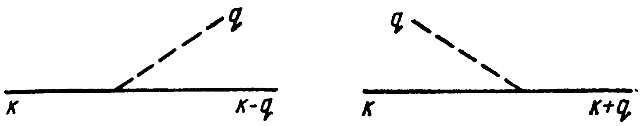
*It should be understood that the "waves" actually represent collective oscillations in which all of the plasma particles participate; however, here, by "particles" we mean only a small group of "resonance" particles which occupy a small volume in phase space, but which exhibit a strong interaction with the "waves."



a b

Fig. 1

$\sqrt{N_q}$ so that the probability for both processes W is the same, being given by

$$W(k, q) = N_q w_{k, k+q} \delta(\varepsilon_i - \varepsilon_f);$$

$$w_{k, k+q} = w_{k+q, k}.$$

As a result of the emission or absorption of a wave, the particle changes its momentum and is transferred to another point in phase space.

The change in the number of particles at point k in phase space is made up of loss terms due to the absorption of plasmons

$$-\sum_q F_k N_q w_{k, k+q} \delta(\varepsilon_k + \hbar\omega_q - \varepsilon_{k+q})$$

and due to the emission of plasmons

$$-\sum_q F_k N_q w_{k, k-q} \delta(\varepsilon_k - \hbar\omega_q - \varepsilon_{k-q})$$

and of gain terms due to the absorption of plasmons

$$+\sum_q F_{k-q} N_q w_{k-q, k} \delta(\varepsilon_{k-q} + \hbar\omega_q - \varepsilon_k)$$

and the emission of plasmons

$$+\sum_q F_{k+q} N_q w_{k+q, k} \delta(\varepsilon_{k+q} - \hbar\omega_q - \varepsilon_k).$$

Here, F_k is the particle distribution function in phase space; ε_k is the kinetic energy of a particle with wave vector k; $\hbar\omega_q$ is the energy of the wave denoted by q.

Summing the contributions of the various processes, we obtain the following equation for the particle distribution function F:

$$\partial F_k/\partial t = \sum_q N_q (\Psi_{k+q, q} - \Psi_{k, q}), \tag{9b}$$

where

$$\Psi_{k,\,q} = (F_k - F_{k-q})\, w_{k,\,k-q}\delta(\varepsilon_k - \varepsilon_{k-q} - \hbar\omega_q).$$

An equation for the wave distribution function N_q can be obtained in similar fashion. The change in N_q occurs as a result of the same processes of emission and absorption of plasmons by particles, so that in the spatially homogeneous case being considered here,

$$\partial N_q/\partial t = N_q \sum_k \Psi^*_{k+q,\,q}. \tag{8b}$$

In order to obtain the equations for a low-density plasma [(8)-(9)] from Eqs. (8b) and (9b),* we take account of the fact that the relative change in the momentum of a particle in the emission (absorption) of a wave in a low-density plasma is always small ($q/k \to 0$) and make use of the following formulas for the probability w and the number of photons associated with the plasma oscillations N_q:

$$w_{k,\,k-q} = 4\pi^2 e^2\omega_0/q^2; \quad N_q = |E_q^2|/4\pi\hbar\omega_0$$

where ω_0 is the plasma frequency. Under these conditions, Eq. (9b) coincides with Eq. (9) and Eq. (8b) becomes the formula for the growth rate (8).

In practice it is easier to obtain the kinetic equation for the plasmon distribution function (for the spectral density of the noise) by solving the linearized kinetic equation with the self-consistent field and determining the growth rate (damping rate) γ; in this case, the quantity γ is a functional of the averaged resonance particle distribution function F against the background of which the small oscillations occur. Thus, in place of Eq. (8b) we have

$$\frac{1}{|E_k^2|}\frac{d|E_k^2|}{dt} = 2\gamma\{F\}. \tag{8c}$$

Equations (8b) and (9b) describe the interaction between the plasmons and particles in a weakly turbulent plasma.

Problem I. Derive Eqs. (8b) and (9b) from the equations for the density matrix of the plasma [20].

Solution. As in the case of the classical plasma, we start from the equation with the self-consistent field φ; in this case we obtain the following

*Equations (8b) and (9b) can be derived from the equations for the density matrix of the plasma (cf. Problem I).

expression for the density matrix in the Wigner representation:

$$f_{xp} = \sum_{\xi} e^{-i\xi p} \varrho \left(x - \frac{\xi}{2}, x + \frac{\xi}{2} \right),$$

where $\rho(y, z)$ satisfies the equation *

$$i\partial\varrho(y, z)/\partial t = [-\Delta_y/2 + \Delta_z/2 + e\varphi(y) - e\varphi(z)]\,\varrho(y, z) =$$

$$= \left[\nabla_x\nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right] \varrho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right),$$

where we have

$$\frac{\partial f_{xp}}{\partial t} = \frac{1}{i} \sum_{\xi} e^{-i\xi p} \left[\nabla_x\nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right] \sum_{q} e^{i\xi q} f_{xq} =$$

$$= -p\frac{\partial f_{xp}}{\partial x} + \frac{1}{i} \sum_{\xi q} \left[e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right] e^{i\xi(q-p)} f_{xq}. \qquad \text{(A)}$$

Equation (A), together with Poisson's† equation

$$\Delta_x\varphi = 4\pi ne \left(\sum_{p} f_{xp} - 1 \right) \qquad \text{(B)}$$

in the quasi-linear plasma theory are replaced by a system of equations for the mean value of the quantum distribution function $f_p^0 = \langle f_{xp} \rangle$ and for the time oscillatory deviation of the distribution function f_{xp} from its mean value (this deviation is assumed to be small).

In Eqs. (A) and (B) we isolate the oscillatory terms and convert to Fourier space components

$$\varphi(x) - \sum_{k} \varphi_k e^{ikx}; \quad f_{xp} - \langle f_{xp} \rangle = \sum_{k} f^1_{kp} e^{ikx},$$

*We take $\hbar = m = 1$ and, for simplicity, consider a quadratically isotropic spectrum $\varepsilon_p = p^2/2$.

†For reasons of simplicity we consider the case of longitudinal oscillations of an electron plasma with a positive space charge background.

thereby obtaining, for the spatially homogeneous case ($\nabla_x f^0 = 0$)

$$\dot{f}^1_{kp} + ikpf^1_{kp} = e\varphi_k \frac{\overset{0}{f}_{p+\frac{k}{2}} - \overset{0}{f}_{p-\frac{k}{2}}}{i}; \quad \varphi_k = -4\pi nek^2 \sum_p f^1_{kp}. \tag{C}$$

On the other hand, carrying out an averaging over x in Eq. (A), we find

$$\frac{\partial f^0_p}{\partial t} = i \sum_k e\varphi^+_k \left[\overset{1}{f}_{k,\, p+\frac{k}{2}} - \overset{1}{f}_{k,\, p-\frac{k}{2}} \right]. \tag{D}$$

Substituting the solution of the ordinary differential equation (C) in (D), and introducing the notation

$$w_{p,\,p'} = 4\pi^2 e^2 \frac{\omega_{p-p'}}{|p-p'|^2}, \quad N_k = \frac{k^2 |\varphi^2_k|}{4\pi\omega_k}, \quad F_p = f^0_p,$$

we have

$$\frac{\partial F_p}{\partial t} = \sum_k w_{p,\,p+k} N_k \left\{ (F_{p-k} - F_p)\, \delta\left(\omega_k - k\left[p + \frac{k}{2}\right]\right) - \right.$$
$$\left. - (F_p - F_{p-k})\, \delta\left(\omega_k - k\left[p - \frac{k}{2}\right]\right) \right\}.$$

Similarly, we find that N_k obeys the equation

$$\partial N_k/\partial t = N_k \sum_p w_{p+\frac{k}{2},\, p-\frac{k}{2}} \left(F_{p+\frac{k}{2}} - F_{p-\frac{k}{2}} \delta(\omega_k - kp) \right).$$

§ 3. Relaxation of Plasma Oscillations

We now wish to consider the damping of plasma oscillations within the framework of the quasi-linear theory. The linear theory predicts an exponential damping in a time of order $1/\gamma$. But the damping rate for this case γ is determined in the linear theory by a thermodynamic equilibrium (Maxwellian) distribution function, since it is assumed that the plasma is in thermodynamic equilibrium when the oscillations are excited. Thus, the infinitesimally small perturbation produced in the plasma decays gradually in accordance with the linear theory, and the system returns to the thermodynamic equilibrium state.

However, if the energy of the initial plasma oscillations is appreciably greater than the energy of the equilibrium thermal noise, the process by which the oscillations are damped is somewhat different. As long as the wave energy density ε is much larger than the thermal energy density nT/H_D, particle collisions are unimportant and the wave diffusion process equalizes the distribution function in the region of phase space that corresponds to resonance particle velocities. As a result of this equalization process particles from low-velocity regions are transferred to regions of higher velocities and the damping of the plasma oscillations is accompanied by an increase in the kinetic energy of the particles (the quantity γ is negative); the total energy of the wave-particle system is conserved in the process. This quasi-linear absorption process is terminated when γ becomes zero. Under these conditions, the energy of the plasma oscillations is finite and appreciably greater than the level of the thermal noise. At this point the oscillations are not longer damped, since $\gamma = 0$ and the distribution function remains unchanged. Subsequently, because of particle collisions, there is a slow diffusion in velocity space which eventually leads to the establishment of a thermodynamic equilibrium (Maxwellian) distribution and the reduction of the oscillations to the thermal noise level; this second stage requires a time interval much longer than the first. In the present section we only consider the first stage, the quasi-linear relaxation of the oscillations.

Let us consider the simplest case of electron plasma oscillations. We assume that at an initial time $t = 0$ in a plasma in thermodynamic equilibrium (the electron velocity distribution is Maxwellian) uniformly in all space over some range of wave vectors k there are produced plasma waves with a spectral energy density $\varepsilon_k(0)$ which is appreciably greater than the thermal noise. All the vectors k are assumed to be parallel to each other, i.e., we are considering a one-dimensional problem. In this case the equations are simplified considerably and an analytic solution can be found. For the one-dimensional spectrum and long wavelengths the velocity of the resonance particles is related uniquely to the wave vector by the simple expression

$$v = \omega_0/k, \tag{10}$$

where ω_0 is the plasma frequency. The coefficient for wave diffusion is then

$$D(v) = \frac{e^2}{2m^2}\frac{|E_k^2|}{v}, \tag{11}$$

while the damping (growth) is given by

$$\gamma = \frac{\pi}{2}\frac{\omega_0^3}{k^2}\frac{\partial f}{\partial v}, \tag{12}$$

where f is the normalized ($\int f\,dv = 1$) average electron distribution function for the velocity component in the direction of the wave vector k.

Thus the system of quasi-linear equations assumes the form

$$\partial\varepsilon/\partial t = A\varepsilon\partial f/\partial v; \tag{13}$$

$$\partial f/\partial t = \frac{\partial}{\partial v}\left(B\varepsilon\frac{\partial f}{\partial v}\right), \tag{14}$$

where $\varepsilon = E_k^2/8\pi$ and f is a function of time t and velocity $v = \omega_0/k$, while the coefficients A and B depend on the velocity but not time:

$$A = \pi\omega_0^2 v^2; \quad B = \omega_0^2/nmv. \tag{14a}$$

The initial conditions for Eqs. (13) and (14) are the following: when $t = 0$ the quantity $\varepsilon = \varepsilon_0(0, v)$, $f = f_M(v)$; here the spectral density $\varepsilon_0(0, v)$ is nonzero in a finite range of velocities $v_1 < v < v_2$, while f_M is the Maxwellian distribution function.

Under the effect of wave diffusion in the region $v_1 < v < v_2$ the negative derivative of the distribution function is increased, i.e., the slope of the distribution function becomes steeper. At the same time, the waves are damped and the diffusion coefficient is reduced. If the initial spectral density of the noise $\varepsilon(0, v)$ is sufficiently large, the value of $\partial f/\partial v$ becomes zero and the noise density $\varepsilon(\infty, v)$ remains finite. Under these conditions, the system goes to a state in which $\partial f/\partial v = 0$ in the range $v_1 < v < v_2$, while $f = f_M$ outside this velocity range. The diffusion coefficient D (and the energy density of the plasma waves) will be nonzero in the region $v_1 < v < v_2$ and zero outside of this region. According to the quasi-linear theory, this state with a plateau in the distribution function should be stationary, since Eqs. (13)-(14) are satisfied in velocity space under these conditions. Actually, as we have already indicated, particle collisions, which are not considered in Eqs. (13)-(14), lead to a slow particle diffusion in velocity space and to the gradual establishment of thermodynamic equilibrium. Thus, the "plateau" distribution described here is quasi-stationary. If slow processes are neglected, however, it can be regarded as stationary.

The equations of the quasi-linear theory (13)-(14) can be used to relate the spectral energy density of the plasma waves $\varepsilon(\infty, v)$ in the stationary state to the initial spectral density $\varepsilon(0, v)$. Substituting $\varepsilon\partial f/\partial v = A^{-1}\partial\varepsilon/\partial t$ from Eq. (13) in Eq. (14), we see that the quantity $f - (\partial/\partial v)BA^{-1}\varepsilon$ is conserved in the relaxation process:

$$\frac{\partial}{\partial t}\left\{f - \frac{\partial}{\partial v} BA^{-1}\varepsilon\right\} = 0. \tag{15}$$

Hence, at any instant of time,

$$f - \frac{\partial}{\partial v} BA^{-1}\varepsilon = f_M - \frac{\partial}{\partial v} BA^{-1}\varepsilon_0.$$

In particular, in the final state (when $t \to \infty$),

$$f_\infty - \frac{\partial}{\partial v} BA^{-1}\varepsilon_\infty = f_M - \frac{\partial}{\partial v} BA^{-1}\varepsilon_0,$$

so that

$$\varepsilon(\infty, v) = \varepsilon(0, v) - AB^{-1}\int_{v_1}^{v} (f_M - f_\infty)\, dv. \tag{16}$$

Since the height of the plateau f_∞ is a known constant (it is determined by the conservation of the total number of resonance particles* $\int_{v_1}^{v_2} (f_M - f_\infty)\, dv = 0$, i.e., $f_\infty = (v_2 - v_1)^{-1} \int_{v_1}^{v^2} f_M\, dv$) the relation in (16) determines the spectral density of the wave energy in the final stationary state.

The reduction in the wave energy as a result of the quasi-linear relaxation process is compensated by the growth in kinetic energy of the particles: as a result of diffusion in phase space there is a net particle transfer to regions of higher velocity. It follows from Eq. (16) that

$$2\int_{\omega/v_2}^{\omega/v_1} \{\varepsilon(0, v) - \varepsilon(\infty, v)\} \frac{dk}{2\pi} =$$

$$= \int_{v_1}^{v_2} dv' nmv' \int_{v_1}^{v'} (f_M - f_\infty)\, dv. \tag{17}$$

*This relation follows from the conservation of the total number of particles: since the distribution function $f(v)$ does not change for $v < v_1$ or $v > v_2$, the total number of resonance particles ($v_1 < v < v_2$) must be conserved.

Integrating the right side of this relation by parts we obtain the energy conservation relation:

$$2 \int [\varepsilon(0, v) - \varepsilon(\infty, v)] \frac{dk}{2\pi} = \int_{v_1}^{v_2} n \frac{mv^2}{2} (f_M - f_\infty)\, dv. \qquad (18)$$

It can be shown that the initial energy of the waves is not sufficient to establish a plateau on the electron distribution [Eq. (16) leads to a meaningless negative expression for ε_∞]. In this case, the stationary state is not. reached and the system goes to thermodynamic equilibrium in a time of the order of the mean time between binary particle collisions.

Up to this point we have been considering a rarefied plasma in which particle collisions can be neglected. In general, the effect of collisions on particle motion will be comparable with the wave effect only when the wave is in equilibrium, i.e., when the wave amplitude is not greater than the amplitude of the corresponding mode in the thermal noise spectrum. For thermal noise, processes such as Cerenkov emission of a wave by the moving particle, emission in collisions, wave Landau damping and collisional absorption, are comparable. Indeed, the level of thermal noise of the plasma waves is determined by the balance between these phenomena.

For high-amplitude waves (superthermal) the particle collisions can still be quite important in certain phenomena, for instance, resonance absorption.

The effect of the wave is to produce a strong distortion of the distribution function in the region of the resonance particles. However, collisions partially restore the Maxwellian distribution function and establish the stationary wave absorption. All other effects due to collisions are negligibly small. Formally, the equation that describes the behavior of the averaged distribution function in time is obtained as the first term in an expansion of the exact kinetic equation in the quantity $1/N_D$ (the ratio of the thermal noise to the thermal energy of the plasma):

$$\frac{df}{dt} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + Sf, \qquad (19)$$

where the last term describes the binary collisions of resonance particles with the other plasma particles.

There are various ways of writing the collision term for a plasma. For example, this term can be written in the form given by Landau [3, 6] and then linearized since the number of resonance particles is small:

$$Sf = L\frac{\partial}{\partial v_i}v^{-3}\left[v_i f + \left(v^2\delta_{ik} - v_i v_k - \frac{T}{m}\,\frac{v^2\delta_{ik} - 3v_i v_k}{v^2}\right)\frac{\partial f}{\partial v_k}\right],$$

where $L = \lambda\omega_0^4/n$ (here λ is the Coulomb logarithm).

However, if we are interested in the particle distribution for only one velocity component $v_{\parallel}$ and integrate the collision term over the other components, we find

$$\int Sf\,dv_{\perp} = \frac{\partial}{\partial v_{\parallel}}\nu\left(v_{\parallel}f + \frac{T}{m}\frac{\partial f}{\partial v_{\parallel}}\right), \tag{19a}$$

where T is the electron temperature while $\nu \approx \lambda\omega_0^4/nv_{\parallel}^3$ is the collision frequency.

Binary collisions between particles lead to the gradual disappearance of the plateau on the distribution function and bring the system to the thermodynamic equilibrium state. The characteristic time for the system to reach the Maxwellian distribution can be estimated as follows. The quasi-linear equation for the distribution function (taking account of diffusion due to emission and absorption of waves and binary collisions) is of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_{\parallel}}D\frac{\partial f}{\partial v_{\parallel}} + \frac{\partial}{\partial v_{\parallel}}\nu\left(v_{\parallel}f + \frac{T}{m}\frac{\partial f}{\partial v_{\parallel}}\right). \tag{20}$$

We now integrate Eq. (20) in time, assuming that $D(\partial f/\partial v) = A^{-1}(\partial D/\partial t)$, where A is given by Eq. (14a), thus obtaining (the subscript on $v_{\parallel}$ is omitted)

$$\left(f - \frac{\partial}{\partial v}A^{-1}D - \frac{\partial}{\partial v}\nu\frac{T}{m}A^{-1}\ln D\right)\Big|_0^t = \frac{\partial}{\partial v}\int_0^t dt\nu v f. \tag{20a}$$

If we consider the case in which the change in the distribution function $f_0 - f_M$ is appreciably smaller than $\partial D_0 A^{-1}/\partial v$, the first term on the left side can be neglected. Then we integrate Eq. (20a) with respect to v from $-\infty$ to v, making use of the fact that the distribution function f is not changed appreciably in the damping process (this is obviously not true for the derivative of this function $\partial f/\partial v$); hence we can replace f under the integral sign on the right side of Eq. (20a) by the thermodynamic equilibrium function $f_M = (2\pi T/m)^{-\frac{1}{2}}\exp - (mv^2/2T)$. Carrying out this procedure, we obtain the following transcendental equation for D(t, v) which gives the time dependence of the diffusion coefficient D (or the wave energy $\varepsilon = B^{-1}D$):

$$\left(-\frac{m}{T\nu}D - \ln D\right)\Big|_0^t = -A\frac{\partial f_M}{\partial v}t.$$

It is evident from this equation that the noise decays in linear fashion in the initial stage $D_0 - D \sim t$, and that the decay only becomes exponential $\ln(D_0/D) \sim t$ in the later stages of the process, when the noise level has become small.

For this reason, the feedback effect of the waves on the particles, which is introduced in the quasi-linear theory, leads to a sharp reduction in absorption: the resonance particles are redistributed and a plateau is formed on the distribution function; however, the collisions gradually smooth the edge of the plateau and a stationary state is established in which

$$\frac{\partial f}{\partial v} = -\frac{\nu v f}{\nu \dfrac{T}{m} + D}.$$

The main effect due to wave feedback is to change the derivative of the distribution function rather than the distribution function itself, so that

$$\frac{\partial f}{\partial v} = \frac{1}{1 + D\dfrac{m}{T\nu}} \frac{\partial f_M}{\partial v} = \frac{1}{1 + \lambda \dfrac{\bar{k}}{\Delta k}\left(\dfrac{v}{v_T}\right)^3 \dfrac{\varepsilon}{nT/N_D}} \frac{\partial f_M}{\partial v}, \tag{21}$$

where $\bar{k}$ is the mean wave number of the packet; Δk is the halfwidth; $v_T = \sqrt{T/m}$; $\lambda \approx 1$.

From Eqs. (13) and (21) we have

$$\frac{\partial D}{\partial t} = D \frac{1}{1 + D\dfrac{m}{T\nu}} \frac{\partial f_M}{\partial v}.$$

Integrating this ordinary differential equation for D(t) (the velocity v appears as a parameter), we obtain Eq. (20b).

Substituting Eq. (21) in Eq. (8), we have (Fig. 2):

$$2\gamma = \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t} = \frac{1}{1 + \lambda \dfrac{\bar{k}}{\Delta k}\left(\dfrac{v}{v_T}\right)^3 \dfrac{\varepsilon}{nT/N_D}} 2\gamma_0. \tag{22}$$

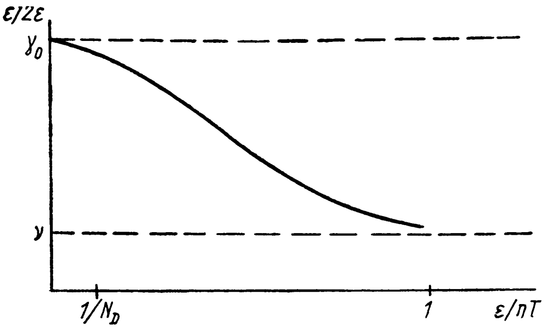


Fig. 2

When $\varepsilon \ll nT/N_D'$ (here N_D' is the number of particles in a sphere of radius v/ω), the quantity $\gamma \to \gamma_0$ where γ_0 is obtained from the linear theory. When $\varepsilon/nT \approx 1$, the quantity $\gamma = (\gamma_0/N_D) \approx \nu$. Hence, the effective damping time for the packet is changed from a quantity of order $1/\gamma_0$, for waves below the thermal noise level, to a time of the order of the collision time, for high-amplitude waves.

The factor that reduces the derivative $\partial f/\partial v$ is a consequence of the distortion of the distribution function and can be written in the form

$$\frac{1}{1 + D\dfrac{m}{Tv}} = \frac{1}{1 + \dfrac{e^2}{mTvk}\dfrac{E_k^2 \Delta k}{\Delta v}}$$

for a "monochromatic" wave (in which $\Delta v \approx \sqrt{e\varphi_0/m}$; $E_k^2 \Delta k = E^2$) this factor is

$$\frac{1}{1 + \dfrac{A'e^2}{mTvk}\dfrac{E^2}{\sqrt{e\varphi_0/m}}} = \frac{1}{1 + A\dfrac{(e\varphi_0)^{3/2}}{\sqrt{m}\,Tv\lambda}}, \qquad (23)$$

where A and A' are approximately equal to 1.

Let us now consider the propagation, through a plasma layer, of plasma waves which are generated continuously at the boundary of the layer.*

*In the framework of the quasi-linear theory we can only consider the propagation of a wave packet with some finite minimum width; this requirement arises since it is necessary to satisfy the condition $\Delta(\omega/k) > \sqrt{e\varphi/m}$, where φ is the mean amplitude of the potential associated with the waves.

The linear theory of small oscillations in a low-density plasma predicts a collisionless damping of waves that propagate in the plasma. In particular, a consequence of this collisionless damping is the attenuation of longitudinal plasma waves that are excited at the boundary of the plasma by an external electric field with frequency $\omega > \omega_0$; these waves are assumed to propagate perpendicularly to the boundary. For plasma waves, which are the only ones we consider, the variation in wave amplitude with distance into the plasma is given by the expression [1]*

$$\varepsilon_k^{-1} \frac{\partial \varepsilon_k}{\partial x} = \frac{\pi}{3} \frac{\omega_0^4}{k^3} \frac{m}{T} \frac{\partial f}{\partial v}, \tag{24}$$

where $k = \omega/v$ is the wave vector; $\omega^2 = \omega_0^2 + 3(T/m)k^2$; f is the electron distribution function for the velocity component parallel to the direction of wave propagation (perpendicular to the boundary). Thus, the linear theory, in which the energy of the wave packet is assumed to be infinitesimally small, leads to exponential damping of the wave packet as a function of distance. The damping factor is given by (24) with

$$f = f_M(v) = (2\pi T/m)^{-1/2} \exp - mv^2/2T.$$

Actually, however, the wave energy is finite and the wave diffusion effect causes an equalization of the distribution function for the resonance electrons with a consequent reduction in damping. If we take account of the fact that the parameter $N_D\varepsilon/nT$ (where ε is the energy density in the wave) is appreciably greater than unity, and neglect collisions, the equations of the quasilinear theory then indicate that the waves will produce a plateau on the distribution function at some distance from the boundary:

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} = 0, \quad D \neq 0;$$

beyond this point is characterized by zero damping:

$$f(v, x) = \text{const}; \qquad \frac{\partial \varepsilon_k}{\partial x} = 0.$$

In order to obtain a finite absorption it is then necessary to introduce a collision term in the equation for the particle distribution function

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + Sf. \tag{25}$$

*This expression (which only holds for distances of the order of several wavelengths away from the boundary) follows from Eq. (8b) since

$$\partial N_q/\partial t \to [H_q, N_q] = 3 \frac{q}{\omega} \frac{T}{m} \frac{\partial N}{\partial x}.$$

Equations (24) and (25) with the initial conditions $\varepsilon(0, v) = \varepsilon_0(v)$ and $f(0, v) = f_0(v)$ determine the spectral energy density of the wave and the distribution function as functions of distance. In order to simplify the calculation, however, we shall limit ourselves to the case of strong waves:

$$\frac{\varepsilon}{nT} \gg \frac{1}{\sqrt{N_D}}.$$

In this case the $v\partial f/\partial x$ term in Eq. (25) can be neglected. Then

$$\frac{\partial f}{\partial v} = -\nu \frac{vf}{D}, \tag{26}$$

where the quantity $(T/m)(\partial f/\partial v)$ is negligibly small compared with vf in the velocity region of interest. If the plasma wave packet is not very broad, the distribution function does not change appreciably (this remark obviously does not apply to the derivative of the distribution function, which can exhibit a substantial change) and f in the right side of Eq. (26) can be replaced by the Maxwellian function f_M. Substituting the value found for $\partial f/\partial v$ in Eq. (24), we then have

$$\frac{\partial \varepsilon_k}{\partial x} = -\frac{\pi}{3}\frac{\nu}{k}\frac{v^2}{T/m} nmv^2 f_M,$$

so that the energy of the wave packet decreases linearly with increasing distance from the boundary:

$$\frac{\varepsilon_k(x)}{\varepsilon_k(0)} = 1 - \frac{x}{L},$$

while the characteristic damping length L is directly proportional to the wave energy at the boundary, being of order

$$L \approx \frac{1}{k}\frac{\varepsilon N_D}{nT} \text{ (when } v \approx v_T).$$

Thus, the quantity L is appreciably greater than the damping length L_{lin} given by the linear theory. For wavelengths of the order of the Debye radius we find $L/L_{lin} \approx \varepsilon N_D/nT$.

The formula for the quasi-linear damping rate of a wave in an anisotropic plasma is complicated, but its general structure is very much the same as that in Eqs. (22)-(23):

$$\gamma = \frac{\gamma_0}{1 + \dfrac{\nu'}{\nu_e}};$$

where ν_e is the electron collision frequency while ν' is the reciprocal time for formation of the quasi-linear plateau.

§ 4. Growth of Perturbations in an Unstable Plasma

Using the quasi-linear theory we now wish to consider the development of a perturbation in an unstable low-density plasma. We shall first investigate the dynamics of a system which is unstable against the excitation of electron plasma oscillations. In order to simplify the problem we consider the case in which the wave vectors characteristic of the growing waves are parallel to each other and in which the wave spectrum is one-dimensional.* We assume that the initial electron distribution function $f(0, v)$ exhibits a rising part in some small range of velocities (the mean velocity in this range is appreciably greater than the mean thermal velocity of the plasma electrons), so that df/dv is positive in this region. Under these conditions the plasma is unstable and the spectral energy density ε_k in the corresponding range of wave numbers $k = \omega_0/v$ starts to grow in accordance with the relation

$$\frac{\partial \varepsilon_k}{\partial t} = 2\gamma\varepsilon_k; \quad \gamma = \frac{\pi}{2}\frac{\omega_0^3}{k^2}\frac{\partial f}{\partial v}. \tag{27}$$

The growing oscillations lead to an increased diffusion coefficient for the resonance particles that interact with the waves:

$$D = \frac{e^2}{2m^2}\frac{|E_k^2|}{v}. \tag{28}$$

Simultaneously, the distribution function is smoothed and the region of instability expands:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v}. \tag{29}$$

The wave growth and the diffusion of resonance electrons continue until a plateau is formed on the distribution function, i.e., a region in which $\partial f/\partial v = 0$. After this point the waves no longer grow and a stationary state is established. The electron distribution in the final state $f_\infty(v)$ can be found from

*This situation holds when a plasma exhibits a preferred direction (external magnetic field, axis of the plasma tube, etc.) and the growth rate is a maximum for wave vectors in this direction.

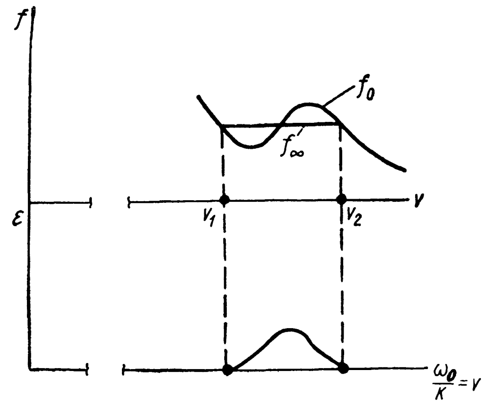


Fig. 3

the conservation of the total number of resonance particles that have diffused to lower velocities in phase space in the establishment of the stationary state:

$$\int_{v_1}^{v_2} f(0, v)\,dv = \int_{v_1}^{v_2} f(\infty, v)\,dv. \tag{30}$$

The velocities $v_{1,2}$ are determined by the boundaries of the plateau region and must be found simultaneously with $f(\infty, v)$ by solving Eq. (30) together with the relation

$$f(0, v_1) = f(0, v_2) = f(\infty); \tag{31}$$

this means that the area under the curves $f(0, v)$ and $f(\infty, v)$ (Fig. 3) must be the same (the points $v_{1,2}$ are shown in the figure).

The system of equations (30)-(31) determines uniquely the value of $f(\infty, v)$ in the plateau region and the boundaries of this region $v_{1,2}$; outside the plateau $f(\infty, v) = f(0, v)$.

As in the quasi-linear relaxation of the plasma oscillations considered in the preceding section, the spectral density of the noise in the final state ε_∞ is related to the initial spectral density ε_0 and the change in the distribution function $f(0, v) - f(\infty, v)$ by the expression

$$\varepsilon(\infty, v) - \varepsilon(0, v) = -AB^{-1}\int_{v_1}^{v} (f_0 - f_\infty)\,dv, \tag{32}$$

where the functions A and B are determined by Eq. (14a). If the initial noise level in the system is thermal, the quantity $\varepsilon(0, v)$ can be neglected, so that the spectrum of plasma waves in the final state is determined only by the initial electron distribution in the vicinity of the growth region f_0 [14, 18]:

$$\varepsilon(v, \infty) = AB^{-1} \int_{v_1}^{v} (f_\infty - f_0)\, dv. \tag{33}$$

The energy density of the waves that are established at the termination of the diffusion process is of order

$$\frac{E^2(\infty)}{8\pi} \approx \delta n \left(mv_2^2 - mv_1^2\right),$$

where $v_{1,2}$ represent the plateau boundaries, while δn is the density of electrons that move to regions of lower energy in velocity space.

The time τ required for excitation of waves and relaxation of the electron distribution (establishment of the plateau) can be estimated from the diffusion time in velocity space by using the expression for the diffusion coefficient D_∞ in the final state:

$$\tau \approx \frac{(v_2 - v_1)^2}{D_\infty} \approx \frac{1}{\omega_0} \frac{(v_2 - v_1)^2}{v^2} \frac{n}{\delta n},$$

where v is the mean value of the velocity in the plateau region.

The development of the instability means that the kinetic energy of the resonance particles is converted into electrostatic energy associated with the plasma waves and into the kinetic energy of all of the plasma electrons, which participate in these collective oscillations; the total energy of the plasma is obviously conserved.

§ 5. Interaction of a Beam with Plasma

It is well known that a system consisting of a plasma and a beam of charged particles that passes through the plasma can be unstable under certain conditions. This so-called electrostatic instability has been the subject of a large number of experimental and theoretical papers. According to the linear theory [4], the electrostatic instability is somewhat different in two limiting cases. When the beam is dense and mono-energetic, and moves with a high velocity with respect to the plasma, the plasma exhibits growing oscillations; the frequency and growth rate are determined by the parameters of the entire system. On the other hand, if the velocity and density of the beam are not very large, and if the velocity spread in the beam is not too small, the frequency of oscillation is equal to the plasma frequency of the plasma and it is only the growth rate that is determined by the properties of the overall system; this growth rate is then proportional to the velocity derivative of the combined distribution function for the plasma electrons and the beam electrons (at the point $v = \omega/k$).

The quasi-linear theory of the earlier sections can be used to investigate the dynamics of a beam–plasma interaction in the second case only.

In analyzing the interaction of a beam with the plasma, as in the preceding sections we shall limit ourselves to one-dimensional electron plasma waves. Assume that the beam moves through the plasma in the positive x direction; at the point $x = 0$ we are given the distribution functions for the electrons in the plasma and in the beam, as well as the spectral density of the noise $\varepsilon_k = |E_k^2|/8\pi$. If particles that are in resonance with the plasma waves ($v = \omega/k$) satisfy the condition $\partial f/\partial v > 0$, the waves will grow. Simultaneously there is a diffusion of the electrons in the beam and plasma in velocity space; this tends to smooth the distribution function in the region in which the wave diffusion coefficient is nonvanishing, thus reducing the growth rate. As the beam continues to move, the velocity derivative of the distribution function diminishes while the wave energy increases. At $x \to \infty$ the system is in the stationary state described above: there is a plateau on the electron distribution function for the beam–plasma system and there is a corresponding region of wave vectors in which there are undamped plasma waves. Since the energy density of these waves is larger than at the input to the system ($x = 0$), it is evident that the kinetic energy of the beam electrons has been reduced. As a result of the formation of a plateau on the electron distribution function, a group of particles has been displaced toward the origin of coordinates in velocity space, indicating a reduction of the kinetic energy of the beam (i.e., the beam is retarded). The quasi-linear theory can be used to find the energy loss of the beam and to determine the shape of the spectrum of plasma waves in the system.

In the case at hand the quasi-linear equations assume the form*

$$\left.\begin{aligned} v_g \frac{\partial \varepsilon}{\partial x} &= A\varepsilon \frac{\partial f}{\partial v}; \\ v \frac{\partial f}{\partial x} &= \frac{\partial}{\partial v} B\varepsilon \frac{\partial f}{\partial v}, \end{aligned}\right\} \qquad (34)$$

*These follow from the general equations (8b) and (9b). However, it is simpler to obtain them from Eqs. (8) and (9) by making the obvious substitutions

$$\frac{\partial \varepsilon}{\partial t} \to \frac{\partial \omega_k}{\partial k}\frac{\partial \varepsilon}{\partial x} = v_g \frac{\partial \varepsilon}{\partial x}; \quad \frac{\partial f}{\partial t} \to \frac{\partial (p^2/2m)}{\partial p}\frac{\partial f}{\partial x} = v\frac{\partial f}{\partial x}.$$

where v_g is the group velocity of the plasma waves, while A and B, which are independent of the x coordinate, are given by Eq. (14a). The system of equations in (34) must be solved with the following boundary conditions: $f(0, v) = f_0(v)$, $\varepsilon(0, v) = \varepsilon_0(v)$. It may be assumed that the wave vector and the velocity of the resonance particles are related by the expression $\omega = kv$, where $\omega = \omega_0 + {}^3\!/_2 k^2 (T/m\omega_0)$, while ω_0 is the electron-plasma frequency and T is the electron temperature.

The level of the plateau formed on the distribution function can be determined from the conservation of the total number of resonance electrons

$$\int_{v_1}^{v_2} f(0, v)\, dv = \int_{v_1}^{v_2} f_\infty\, dv,$$

so that

$$f(\infty) = (v_2 - v_1)^{-1} \int_{v_1}^{v_2} f(0, v)\, dv. \tag{35}$$

Here, $v_{1,2}$ are points in velocity space which define the boundaries of the plateau; the values $v_{1,2}$ can be found simultaneously with f_∞ by solving Eq. (35) together with the equation

$$f(0, v_1) = f(0, v_2) = f_\infty.$$

The spectral energy density of the plasma waves at $x \to \infty$ can be determined as follows. The value of $\varepsilon \partial f/\partial v$ from the first equation in (34) is used in the second, yielding

$$\frac{\partial}{\partial x}\left(vf - \frac{\partial}{\partial v} BA^{-1} v_g \varepsilon\right) = 0,$$

i.e.,

$$\varepsilon(v, \infty) = \varepsilon(v, 0) + AB^{-1} v_g^{-1} \int_{v_1}^{v} v\,(f_\infty - f_0)\, dv. \tag{36}$$

Thus, the development of the two-stream instability and the smearing of the peak in the electron velocity distribution cause some of the kinetic energy of the beam electrons to be converted into plasma wave energy. Obviously the total energy flux remains constant; this can be shown as follows. Consider the case in which the noise level at the input to the system is thermal: $\varepsilon(v, 0) = 0$. Multiplying both sides of Eq. (36) by $2v_g$, and integrating

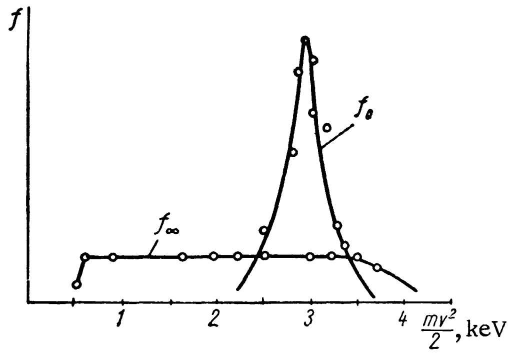


Fig. 4

with respect to wave number $k = \omega/v$, we find that the energy flux through the cross sections $x = 0$ and $x = \infty$ are the same:

$$\sum_k v_g 2\varepsilon_k + \int_{v_1}^{v_2} v \frac{mv^2}{2} (f_\infty - f_0)\, dv = 0. \tag{37}$$

The limits of integration in Eq. (37) can be extended to infinity because the spectral density $\varepsilon(\infty)$ is zero outside the range $v_1 < v < v_2$, and the functions f_0 and f_∞ are identical.

In conclusion, we note that the theoretical conclusions concerning the relaxation of an unstable plasma toward a state with a plateau on the distribution function have been observed experimentally (Fig. 4) [16, 21, 25].

Problem 2. Investigate the development of perturbations when the boundaries of the instability region are fixed (Fig. 5).

Solution. Since $\partial f/\partial v \to -\infty$ at two points $v_0 \pm u$ (Fig. 5) these points represent the boundaries of the instability region and the distribution function can only change for $-u < v - v_0 < u$. If $f(0, v)$, the initial electron distribution, is smooth in this region, it can be expanded in a series in which we limit ourselves to the first two terms:

$$f(0, v) = \text{const} + A_0 (v - v_0).$$

It then follows from the conservation of the number of particles that the constant in this expression is $f(\infty)$.

It can be easily shown that Eqs. (13) and (14) have the solution

$$F(v,\ t) = f(\infty) + A(t)(v - v_0);$$

$$D(v,\ t) = \frac{u^2 - (v - v_0)^2}{2} B(t),$$

where

$$A(t) = \frac{A_0 - B_0}{1 + \frac{B_0}{A_0} \exp(A_0 + B_0)\, t}; \quad B(t) = A_0 + B_0 - A(t).$$

Here, $A_0 = \partial f(0,\ v)/\partial v$, while the quantity B_0 is proportional to the amplitude of the initial noise.

In the solution given here, the distribution functions remains linear throughout the entire process, while the spectrum is parabolic.

Problem 3. Find the spectrum for ion-acoustic waves excited by an electric current in a weakly ionized plasma.

Solution. In the stationary state the equation for the averaged electron distribution function in the resonance region of interest f is given by

$$\frac{-eE_0}{m} \frac{\partial f}{\partial v} + \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + Sf = 0 \qquad \text{(A)}$$

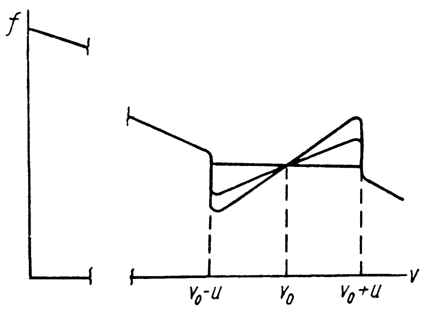


Fig. 5

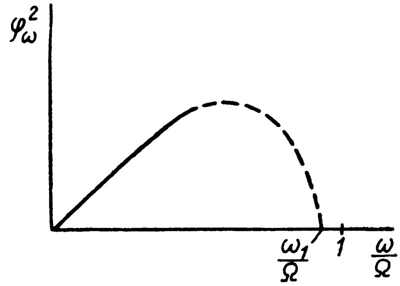


Fig. 6

where E_0 is the external electric field and Sf is the collision term.

The equation for the waves reduces to an equality for the rate of production of waves by electrons and the absorption due to collisions of ions with neutrals [4]

$$\frac{\pi}{m}\frac{\partial f}{\partial v} = \frac{2}{M}\frac{\nu_i}{v^3 k}, \tag{B}$$

where m and M are the masses of the electron and ion, respectively; ν_i is the ion-neutral collision frequency, and k is the wave number. The shape of the spectrum in the low-frequency region can be found as follows. Since

$$D = \frac{\pi e^2}{m^2}\sum_k E_k^2 \delta(\omega_k - kv) \sim \frac{E_k^2}{v - v_g}$$

(where v_g is the group velocity), and since (A) and (B) indicate that for small k

$$D \sim \left(\frac{\partial f}{\partial v}\right)^{-1} \sim k,$$

it then follows that

$$E_k^2 \sim (v - v_g)\, k. \tag{C}$$

Substituting the expressions for $v = \omega_k / k$ and $v_g = d\omega_k / d_k$ in (B), we find the dispersion equation

$$\left(\frac{\omega_k}{\Omega}\right)^2 = \frac{(kR_D)^2}{1 + (kR_D)^2}$$

where Ω is the ion-plasma frequency and R_D is the Debye radius; the noise density in the low-frequency region of the spectrum is then

$$E_k^2 \sim \left(\frac{\omega}{\Omega}\right)^3, \quad (\omega \ll \Omega),$$

and the spectral density of the square of the potential is

$$\varphi_\omega^2 \sim \frac{E_k^2}{k^2 v_g} \sim \frac{\omega}{\Omega}.$$

Thus, in the weakly turbulent state, the quantity φ_ω^2 increases linearly with frequency ω when $\omega \ll \Omega$ (Fig. 6).

At still higher frequencies, the quantity φ_ω^2 reaches a peak and is then reduced, reaching zero when $\omega = \omega_1 \sim \Omega$.

§ 6. Threshold for Wave Absorption in a Plasma and Turbulent Heating

If one considers the propagation of a wave in a plasma at an amplitude exceeding some given threshold value (depending on the type of wave and its period), in certain cases the plasma can be unstable. When this happens, part of the ordered energy of the wave is converted into the energy associated with the spectrum of the nonequilibrium plasma oscillations.

In order to illustrate the effect we consider the excitation of high-amplitude, one-dimensional, ion-acoustic waves in a plasma. It is assumed that the waves cause the electrons to acquire a mean velocity U (with respect to the ions which are at rest), and that this mean velocity is greater than the critical velocity $c_s \approx \sqrt{T_e/M}$. Under these conditions, the ion-acoustic waves grow, causing electron diffusion in velocity space by virtue of wave diffusion. As a result of the equalization of the electron distribution function, the region of instability expands in velocity space and soon encompasses the entire range of allowed values of phase velocities for the ion-acoustic waves $c_i < v < c_s$ ($c_i \approx \sqrt{T_i/M}$).

If a wave of sufficiently high amplitude propagates in the plasma the electron distribution periodically passes through the region $-c_s < v < c_s$ in which the ion-acoustic waves are excited; hence, the quasi-linear diffusion coefficient in velocity space D is different from zero. The electron distribution function will gradually be smoothed and will exhibit a plateau after a period of time (Fig. 7):

$$f(\infty, v) = \begin{cases} f_\infty = \text{const}; & (|v| < U); \\ f_0(v); & (|v| > U), \end{cases}$$

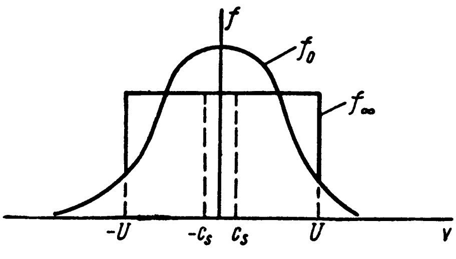


Fig. 7

where U is the maximum displacement of the electron distribution in velocity space caused by the external field (assuming that $U \gg c_s$), while $f_\infty = (1/2U) \times \int_{-U}^{U} f_0(v)\, dv$. The smoothing of the distribution function is described by the diffusion equation

$$\frac{\partial f}{\partial t} = k \frac{\partial^2 f}{\partial v^2}$$

[with the boundary condition $(\partial f/\partial v)_{\pm u} = 0$], where $k \approx \omega c_s^2$ for $U > c_s$; $k = 0$ for $U < c_s$; c_s is the width of the region in which the superthermal noise is excited and ω is the frequency of the external field. Thus, the smoothing time is given by $\tau \approx (U^2/k) \approx (1/\omega)(U/c_s)^2$.

Hence, an external field whose intensity is greater than the threshold value will displace electrons in velocity space and do work; by exciting collective oscillations of the plasma this field can provide "collisionless" heating of the electrons.

Turbulent heating of electrons in an unstable plasma, in which the electrons move with respect to the ions, is characterized by intense high-frequency oscillations in the plasma; the electrons then execute random motions in these high-frequency fields [23]. The presence of these oscillations tends to stabilize the system: for example, assume that high-frequency oscillations are excited in a plasma consisting of a cold electron gas (at rest) and a moving cold ion gas; then the plasma will be stable if the random velocity of the electrons $\sqrt{\langle v^2 \rangle}$ (due to the oscillations) is greater than the relative velocity of the ions with respect to the electrons U_0 (cf. Problems 4-5).

Problem 4. Derive equations to describe slow processes in a plasma in which plasma waves are excited.

Solution. In the presence of the high-frequency waves the plasma particles are subject to a force

$$f = -\sum_k \frac{e^2}{4m\omega_k^2} \nabla E_k^2. \tag{A}$$

The slow (compared with the characteristic period of the high-frequency waves $<1/\omega>$) variations of the distribution function F are described by the following equation (in the absence of external forces):

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{f}{m}\frac{\partial F}{\partial v} = 0. \tag{B}$$

The next equation we seek is the equation for the spectral energy density in the high-frequency waves; the variation in spectral density is given by the following equation for $N_k = E_k^2/\omega_k$:

$$\frac{\partial N_k}{\partial t} + v_g\frac{\partial N_k}{\partial x} - \frac{\partial \omega_k}{\partial x}\frac{\partial N_k}{\partial k} = 0 \tag{C}$$

(v_g is the group velocity).

Problem 5. Investigate the stability of an ion stream moving through a cold electron gas in which plasma waves are excited.

Solution. We compute the first two moments of (B) in Problem 4, obtaining

$$\frac{\partial n}{\partial t} + \frac{\partial nU}{\partial x} = 0; \tag{A}$$

$$\frac{dU}{dt} = \frac{f}{m}, \tag{B}$$

where f is given by (A) of Problem 4.

Linearizing the first of these two relations and (B) of Eq. (4), we find δN_k, the change in the quantity N_k, in a plane wave in which all quantities are proportional to $e^{-i\Omega t + iqx}$:

$$-i\Omega\delta N_k - kiqU\frac{\partial N_k}{\partial k} - \frac{iq\omega_0}{2}\frac{\delta n}{n}\frac{\partial N_k}{\partial k} = 0;$$

$$-i\Omega\delta n + iqnU = 0,$$

i.e.,

$$\delta N_k = -\frac{\delta n}{n}\left(k + q\frac{\omega_0}{2\Omega}\right)\frac{\partial N_k}{\partial k}.$$

The change in the spectral density of the high-frequency waves is

$$\delta E_k^2 = \delta(\omega_k N_k) = \omega_k \delta N_k + N_k\left(\frac{\omega_0}{2}\frac{\delta n}{n} + kU\right) =$$

$$= \frac{\delta n}{n}\left(1 + 2\frac{k}{q}\frac{\Omega}{\omega_0}\right)\left(\frac{\omega_0 N_k}{2} + \frac{\omega_0}{\Omega} q \frac{\partial N_k}{\partial k}\right).$$

Substituting this value of δE_k^2 in (B) (considering, for simplicity, the case $\bar{k} = \sum_k kN_k / \sum_k N_k = 0$), we have

$$\frac{dU}{dt} = -iq\frac{\delta n}{n}\sum_k \frac{e^2}{4m^2\omega_0^2} E_k^2. \tag{C}$$

It is evident from (C) that the presence of the waves is equivalent to the presence of a thermal velocity spread for the electrons (i.e., an electron pressure):

$$\langle v^2 \rangle = \sum_k \frac{e^2}{4m^2\omega_0^2} E_k^2. \tag{D}$$

Hence, from the linear theory of stability of an ion stream moving with velocity U_0 through an electron gas [4], we find that the plasma is stable when

$$U_0 < \sqrt{\langle v^2 \rangle},$$

where $\langle v^2 \rangle$ is given by (D).

§ 7. Plasmon – Plasmon Interactions

Up to this point we have been considering a weakly turbulent plasma, assuming that the wave energy density is small enough so that interaction between waves could be neglected; in this case the important processes are the emission and absorption of collective plasmon oscillations by resonance particles. When the wave energy increases, the interaction between waves becomes important; since many waves are excited simultaneously in a turbulent plasma, and since their phases are random, the interaction between waves reduces to wave "collisions" and can be described on the basis of a kinetic equation for the wave distribution function (plasmon equation) in phase space.*

It is convenient to derive the wave kinetic equation starting with the Lagrangian expanded in powers of the amplitude of the collective plasma oscillations. The complete Lagrangian L for a plasma can be written in the following form (cf. [8]):

$$L = \sum_{\nu} \iint dx\, dv\, f_{\nu} \left\{ \frac{m_{\nu}(v + D_{\nu}y_{\nu})^2}{2} - e_{\nu}V_0(x + y_{\nu}) - \right.$$
$$\left. - e_{\nu}\varphi(x + y_{\nu}) + e_{\nu}(v + D_{\nu}y_{\nu})(A_0(x + y_{\nu}) + \alpha(x + y_{\nu})) \right\} +$$
$$+ \frac{1}{8\pi} \int dx \{ (E_0 + e)^2 - (B_0 + b)^2 \}. \tag{38}$$

Here, $y_{\nu}(x, v, t)$, φ, and α are variables in terms of which the variation is taken, while $f_{\nu}(x, v)$ is the stationary distribution function for particles of species ν in the stationary fields $E_0 = -\nabla V_0$ and $B_0 = \nabla \times A_0$, which satisfy the Maxwell equations

$$\left.\begin{aligned} &\nabla \times E_0 = 0; \quad \nabla \cdot E_0 = \sum_{\nu} \int e_{\nu} f_{\nu}(x, v)\, dv; \\ &\nabla \cdot B_0 = 0; \quad \nabla \times B_0 = \frac{4\pi}{c} \sum_{\nu} \int e_{\nu} v f_{\nu}(x, v)\, dv. \end{aligned}\right| \tag{39}$$

The operator D_{ν} in Eq. (38) is the total time derivative along the trajectory of particles of species ν in the fields E_0 and B_0:

$$D_{\nu} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + a_{0\nu}\frac{\partial}{\partial v}, \tag{40}$$

where

$$a_{0\nu} = \frac{e_{\nu}}{m_{\nu}} \left\{ E_0(x) + \frac{v}{c} \times B_0(x) \right\}. \tag{41}$$

The quantities y, φ, and α represent the displacement of the particles and the deviations of the scalar and vector potentials from equilibrium values in the stationary state. Expanding the Lagrangian L in powers of y, φ, and α we obtain the Lagrangians for the zeroth, first, second, etc., orders L_0, L_1, L_2, ... The zero-order Lagrangian L_0 does not contain y, φ, and α; the

*The interaction between plasmons is analogous to the interaction between phonons in condensed media and the kinetic equations for the two cases are analogous.

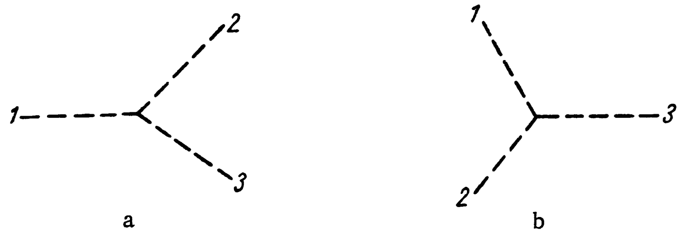


Fig. 8

Lagrangian L_1 vanishes identically; the Lagrangian L_2 describes the harmonic oscillations of the plasma [8]. The third- and fourth-order Lagrangians L_3 and L_4 describe the interaction between these harmonic oscillations (plasmons).

For reasons of simplicity we shall first limit ourselves to the analysis of longitudinal waves in a uniform isotropic plasma ($E_0 = B_0 = 0$); in this case, Eq. (38) yields

$$L_3 = -\sum_\nu \frac{e_\nu}{2} \iint dx\, dv f_\nu(x, v)\, y^\nu_\alpha y^\nu_\beta \nabla_\alpha \nabla_\beta \varphi; \tag{42}$$

$$L_4 = -\sum_\nu \frac{e_\nu}{6} \iint dx\, dv f_\nu(x, v)\, y^\nu_\alpha y^\nu_\beta y^\nu_\gamma \nabla_\alpha \nabla_\beta \nabla_\gamma \varphi. \tag{43}$$

§ 8. Three-Plasmon Processes

We shall first consider interactions between waves in which three plasmons participate. These processes are (a) decay of a single plasmon into two plasmons, and (b) combination of two plasmons into one (Fig. 8).

In these processes we must satisfy frequency conservation ω and wave-vector conservation k (otherwise, the transition probability amplitude vanishes); for cases (a) and (b) these conservation relations are

$$\text{a)}\ k_1 = k_2 + k_3;\ \omega_1 = \omega_2 + \omega_3; \tag{44}$$

$$\text{b)}\ k_1 + k_2 = k_3;\ \omega_1 + \omega_2 = \omega_3. \tag{45}$$

In an isotropic plasma in which there are two kinds of plasmons – the ion-acoustic plasmons (s) and the plasma (longitudinal) (l) oscillations (we assume that the necessary condition of weakly damped ion-acoustic waves is satisfied; the electron pressure exceeds the ion pressure and only weakly damped longwave plasma oscillations are considered), the conservation relations [(44)-(45)] allow only those three-plasmon processes in which two plasma plasmons and one acoustic plasmon participate. The longwave l-plasmons have approximately the same frequency $\omega \approx \omega_{0e}$, so that a single l-plasmon cannot split into two; conversely, the two l-plasmons cannot combine into one. The ion-acoustic oscillations cannot interact between themselves in a three-plasmon process because their spectrum is a "nondecay" type — the frequency of these oscillations ω increases with wave number k at a slower rate than linear (cf. Problem 6). Finally, three-plasmon processes in which two s-plasmons and one l-plasmon participate are not possible in a plasma with a small ratio of electron mass m to ion mass M because the maximum possible frequency of the ion-acoustic wave is much smaller than the frequencies of the plasma waves and the frequency conservation relation cannot be satisfied.

We then consider only the allowed three-plasmon processes, in which one s-plasmon and two l-plasmons participate: we express the displacement y^e in the Lagrangian in (42) in terms of the potential φ (the ion term in L_3 can be neglected since its contribution is small if $m/M \ll 1$, in which case the ion velocity and displacement are small compared with the electron velocity and displacement):

$$y^e = \sum_k \frac{e}{m} ik \left\{ \frac{\varphi_k^s}{(kv)^2} + \frac{\varphi_k^l}{-\omega_{0e}^2} \right\} e^{ikx}, \tag{46}$$

where φ_k^s and φ_k^l are the spatial Fourier components of the potential in the ion-acoustic wave and plasma waves, respectively. Substituting this value of y^e in Eq. (42), we obtain the following expression for the Lagrangian for three-plasmon processes in an isotropic plasma:

$$L_3 = \sum_{p+q+r=0} \Lambda_{p;\,qr} \varphi_p^s \varphi_q^l \varphi_r^l, \tag{47}$$

where

$$\Lambda_{p;\,qr} = \frac{e^3}{2m^2} \int \frac{f dv}{(n \cdot v)^2} \frac{q \cdot r}{2\omega_{0e}^2} \approx \frac{eq \cdot r}{T}. \tag{48}$$

Starting with the classical Lagrangian (47) by introducing second quantization we can write a system of kinetic equations for the distribution functions for the l- and s-plasmons. For the distribution function of the l-plasmons n_k we find

$$\dot{n}_1 = \sum w_{12} \{ -n_1 (n_3 + 1)(N_2 + 1) + (n_1 + 1) n_3 N_2 \} + \\ + \sum w_{21} \{ -n_1 (n_3 + 1) N_2 + (n_1 + 1) n_3 (N_2 + 1) \}, \tag{49}$$

where w_{12} is the probability for decay of an l-plasmon k_1 into an s-plasmon k_2 and an l-plasmon k_3, while w_{21} is the probability for combination of an l-plasmon k_1 and an s-plasmon k_2 into another l-plasmon k_3. The summation in Eq. (49) is taken over wave numbers k_2 and k_3, which satisfy the conservation relation (44) in the first collision integral and the conservation relation (45) in the second collision integral.

The analogous equation for the distribution function for the s-plasmons N_k is

$$\dot{N}_2 = \sum w_{21} \{-N_2 n_1 (n_3 + 1) + (N_2 + 1)(n_1 + 1) n_3\}. \tag{50}$$

Here, w_{21} is the probability for combination of an s-plasmon k_2 and an l-s plasmon k_1 into an l-plasmon k_3. The second collision integral vanished in Eq. (50), since the decay of a low-frequency s-plasmon into two high-frequency l-plasmons is forbidden by the frequency conservation relation (as is the inverse process, combination of two l-plasmons into one s-plasmon).

The probabilities w in (49) and (50) can be expressed in terms of the matrix elements of the Lagrangian (42):

$$w_{12} = \frac{2\pi}{\hbar} |\langle n_1, N_2, n_3 | L_3 | n_1 - 1, N_2 + 1, n_3 + 1 \rangle|^2;$$

$$w_{21} = \frac{2\pi}{\hbar} |\langle n_1, N_2, n_3 | L_3 | n_1 - 1, N_2 - 1, n_3 + 1 \rangle|^2;$$

so that

$$w_{12} = w_{21} = \frac{2\pi}{\hbar} |\Lambda_{p;\, qr}|^2 |\varphi^s_{k_2}|^2 |\varphi^l_{k_1}|^2 |\varphi^l_{k_3}|^2. \tag{51}$$

Here, φ^s and φ^l are the matrix elements for the potential for the s and l waves:

$$\varphi^s_k = \frac{1}{k} \sqrt{2\pi\hbar\omega_s(k)}; \tag{52}$$

$$\varphi^l_k = \frac{1}{k} \sqrt{2\pi\hbar\omega_l(k)}. \tag{53}$$

Substituting the value of w_{21} (51) in the kinetic equation for the s-plasmons, we have

$$\dot{N}_2 = \int \frac{dk_1}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \delta(k_1 + k_2 - k_3)\, \delta(\omega_1 + \omega_2 - \omega_3) \frac{2\pi}{\hbar^2} |\Lambda_{k_2;\, k_1 k_3}|^2 \times$$

$$\times \frac{2\pi\hbar\omega_s(k_2)}{k_2^2} \frac{2\pi\hbar\omega_l(k_1)}{k_1^2} \frac{2\pi\hbar\omega_l(k_3)}{k_3^2} \{-N_2 n_1 (n_3 + 1) +$$

$$+ (N_2 + 1)(n_1 + 1) n_3 \}. \tag{54}$$

Writing $n_k = | E_k^l |^2/\hbar \omega_l$ in this equation, we find the order of the characteristic "collision frequency" of s-plasmons with l-plasmons

$$\gamma_3 = \frac{\dot{N}_2}{N_2} \approx \omega_3 \frac{| E^l |^2}{nT} \tag{55}$$

[the expression for γ_3 can conveniently be written in the form

$$\gamma_3 \approx \omega_s \left(\frac{v_\sim}{U} \right)^2,$$

where $v_\sim$ is the random velocity of the electrons in the oscillations, while U is the phase (or group) velocity for waves characterized by $k \lessgtr R_D^{-1}$].

The quantity γ_3 determines a number of the characteristic features of a weakly turbulent plasma: for example, the characteristic decay length L of the ion-acoustic waves in a plasma with highly excited plasma oscillations is approximately equal to the mean free path for collisions of an s-plasmon with l-plasmons, and is thus related to the collision frequency γ_3 by the expression

$$L \approx c_s/\gamma_3. \tag{56}$$

After substitution of the value of the transition probability w (51), the kinetic equation (50) for the distribution function of l-plasmons assumes the form

$$\dot{n}_1 = \int \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{2\pi}{\hbar^2} | \Lambda_{k_2;\, k_1 k_3} |^2 \frac{2\pi\hbar\omega_s(k_2)}{k_2^2} \times$$

$$\times \frac{2\pi\hbar\omega_l(k_1)}{k_1^2} \frac{2\pi\hbar\omega_l(k_3)}{k_3^2} \cdot \{ (-n_1(N_2+1)(n_3+1) +$$

$$+ (n_1+1) N_2 n_3) \delta(k_1 - k_2 - k_3) \delta(\omega_1 - \omega_2 - \omega_3) +$$

$$+ (-n_1 N_2 (n_3+1) + (n_1+1)(N_2+1) n_3) \delta(k_1 + k_2 - k_3) \times$$

$$\times \delta(\omega_1 + \omega_2 - \omega_3) \}. \tag{57}$$

The system of equations in (54) and (57) determines completely the dynamics of a weakly turbulent isotropic plasma in the absence of resonance particles. When there are resonant particles, it is then necessary to use a sys-

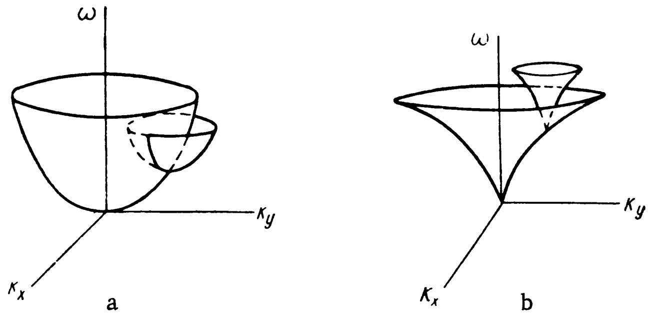


Fig. 9

tem of equations for the waves and resonant particles which takes account of the interaction of the waves (plasmons) in the form of the collision integrals in (54) and (57). It should be noted that the structure of the collision integrals (54) and (57) is simplified to some extent in a classical plasma. Since the mean "population number" N_k is related to the spectral energy density of the oscillations Q_k by $N_k = Q_k/\hbar\omega_k$,* in a classical plasma ($\hbar \to 0$), in treating the collision integral we need only retain the highest-order terms in N or n; in this case, the collision integrals (54) and (57) are quadratic in N and n:

$$-n_1N_2(n_3+1)+(n_1+1)(N_2+1)n_3 \underset{\hbar\to 0}{\to} -N_2n_1+N_2n_3+n_1n_3, \quad (58)$$

$$-n_1(N_2+1)(n_3+1)+(n_1+1)N_2n_3 \underset{\hbar\to 0}{\to} -n_1N_2-n_1N_3+N_2n_3 \quad (59)$$

and after the substitution $N_k = Q_k/\hbar\omega_k$, Planck's constant disappears from (54) and (57).†

*For example, for longwave plasma oscillations,

$$Q_k = \frac{nmv_k^2}{2}+\frac{E_k^2}{8\pi}=\frac{E_k^2}{4\pi}; \quad N_k=\frac{E_k^2}{4\pi\hbar\omega_0}.$$

†The direct derivation of the classical ($\hbar = 0$) collision integrals for plasmons using the hydrodynamic equations has been given in [27, 28].

The kinetic equations that describe three-plasmon processes in an anisotropic (and inhomogeneous) plasma can be obtained similarly by isolating third-order terms in the oscillation amplitude in the Lagrangian [Eq. (38)]. These three-plasmon processes are responsible for a number of important features of a turbulent plasma: in particular, the turbulent transport coefficients for matter, momentum, and energy. A knowledge of these coefficients is required to solve a number of problems: for instance, the structure of the turbulent front of a shock wave in a rarefied plasma [27, 28], or the evaluation of the "anomalous" diffusion coefficient (Problem 7), etc.

Problem 6. What functional relation $\omega = \omega(k)$ must be displayed by the dispersion relation to satisfy the frequency and wave-number conservation relations for three-plasmon interactions between plasmons of one kind?

Solution. For clarity we consider the case in which the frequency ω depends only on the modulus of the two-dimensional wave vector $\mathbf{k} = \{k_x, k_y\}$; in this case, the function $\mathbf{k} = k_x k_y$ represents a surface of rotation about the ω axis in $(k_x k_y \omega)$ space (Fig. 9).

The conservation relations allow three-plasmon processes if the equation $\omega(\mathbf{k}) + \omega(\mathbf{q}) = \omega(\mathbf{k} + \mathbf{q})$ has a solution, i.e., if the ω surface intersects a similar surface but drawn in a coordinate system whose origin lies on the ω surface (Fig. 9a); if this condition is not satisfied, the three-plasmon processes are forbidden (Fig. 9b). It is evident from Fig. 9 that the three-plasmon interactions are forbidden for spectra in which ω increases more slowly than k.

Problem 7. Estimate the value of the "anomalous" diffusion coefficient in a weakly inhomogeneous rarefied plasma in a magnetic field.

Solution. The origin of the anomalous diffusion is the excitation of drift waves in the unstable inhomogeneous plasma [24]. These waves are emitted by electrons and absorbed by ions; as a result there is a transfer of momentum, i.e., a frictional force between the electron and ion gases

$$f \approx \gamma' N \hbar k_{\perp} = \nu_{ef} nmU. \tag{A}$$

Here, $N \approx nMv_{\sim}^2/\hbar\omega$ is the density of the gas of drift waves; ω and $k_{\perp}$ are the characteristic frequency and wave vector, U is the drift velocity, γ' is the frequency of emission of waves by the electrons (the growth rate in the linear theory).

In the stationary turbulent state, the frequency of emission of waves γ' must be equal to the frequency of collisions between waves in three-plasmon processes; when $k_{\perp}\rho_i \approx 1$,

$$\gamma' \approx \frac{k_{\perp}^2 v_{\sim}^2}{\omega}. \qquad \text{(B)}$$

Taking the value of $v_{\sim}^2$ from (B) and substituting in (A), we find the effect of collision frequency ν_{ef} and the coefficient of anomalous diffusion

$$D_a \approx \frac{\gamma'^2}{\omega\omega_{Hi}} \frac{cT}{eH} \approx \frac{m}{M\beta} \frac{\varrho_i}{a} \frac{cT}{eH}.$$

§ 9. Higher-Order Processes

In a number of cases the dispersion relation for the collective plasma oscillations is such that the three-plasmon interactions are forbidden by the frequency and wave-vector conservation relations. In this case it is necessary to consider processes in which four waves participate.

The matrix elements for four-plasmon processes (consequently, the transition probabilities) can be obtained by means of the Lagrangians L_3 and L_4 as in the three-plasmon processes (the contribution to the probability for four-plasmon processes is associated with matrix elements of first order in the perturbation theory in L_4 and second-order in the perturbation theory in L_3).

In the case of an isotropic plasma, which we consider below, the conservation relations forbid three-plasmon processes in which only plasma waves or ion-acoustic waves participate. For this reason the electron plasma waves excited in the plasma are described by an equation which considers the interaction of four plasma waves in addition to the three-plasmon processes considered above. However, if ion-acoustic waves are excited in the plasma, the kinetic situation is described by the interaction of four of these waves.

In general, in the interaction of four plasmons, one can find the following processes: (a) conversion of two plasmons into two other plasmons; (b) decay of one plasmon into three plasmons; (c) combination of three plasmons into one (Fig. 10a, b, c). Hence, the kinetic equation for waves in which four-plasmon processes occur is

$$\frac{dN_1}{dt} = \sum w_{2,2} \{(N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)\} +$$
$$+ \sum w_{1,3} \{(N_1+1)N_2N_3N_4 - N_1(N_2+1)(N_3+1)(N_4+1)\} +$$
$$+ \sum w_{3,1} \{(N_1+1)(N_2+1)(N_3+1)N_4 - N_1N_2N_3(N_4+1)\}. \qquad (60)$$

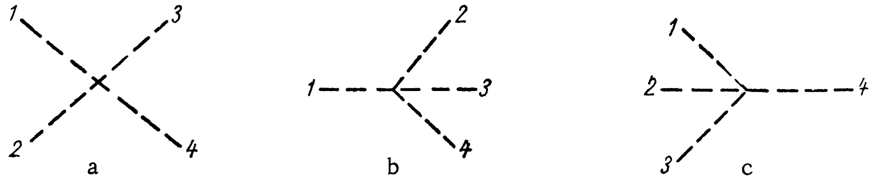


Fig. 10

The summation in Eq. (60) is taken over the wave numbers k_2, k_3, k_4, taking account of the conservation relations which, for the first, second, and third terms on the right side of Eq. (60) can be written as follows:

$$\text{a) } k_1 + k_2 = k_3 + k_4;\quad \omega_1 + \omega_2 = \omega_3 + \omega_4;$$

$$\text{b) } k_1 = k_2 + k_3 + k_4;\quad \omega_1 = \omega_2 + \omega_3 + \omega_4;$$

$$\text{c) } k_1 + k_2 + k_3 = k_4;\quad \omega_1 + \omega_2 + \omega_3 = \omega_4.$$

The kinetic equation for the ion-acoustic waves only contains three terms; as far as the longwave electron plasma oscillations are concerned, we find that processes (b) and (c) are forbidden (since the frequency for these processes is approximately the same, ω_{0e}), so that Eq. (60) assumes the form

$$\dot{N}_1 = \sum w_{2,2}\{(N_1+1)(N_2+1)N_3N_4 - - N_1N_2(N_3+1)(N_4+1)\}. \qquad (61)$$

Thus, the collision integral for four-plasmon processes is

$$\dot{N}_1 = \int B\{(N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)\} \times \times \delta(\omega_1+\omega_2-\omega_3-\omega_4)\, dk_2\, dk_3, \qquad (62)$$

where $k_4 = k_1 + k_2 - k_3$, and

$$B \approx \frac{\hbar^2 k^4}{n^2 m^2}. \qquad (63)$$

Substituting $N_k = E_k^2/\hbar\omega_k$ from Eq. (62), we find the characteristic collision frequency in the gas of plasma waves:

$$\gamma_4 = \frac{\dot{N}}{N} \approx \omega\left(\frac{E^2}{nT}\right)^2, \qquad (64)$$

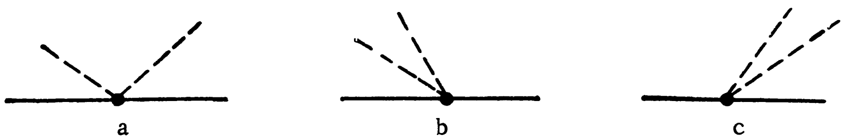


Fig. 11

where $\omega \approx \omega_{0e}$, while $\frac{E^2}{4\pi} = \sum_k E_k^2/4\pi$ is the energy density of the waves.

An estimate of the collision frequency γ_4 in the gas of these waves can be obtained as follows. Since the collision integral is proportional to N^3, the frequency γ_4 must be proportional to N^2, i.e., the fourth power of the ratio of the random velocity of the electrons in the plasma waves to the phase (or group) velocity U for these waves (with wave number $k < 1/R_D$). The coefficient of proportionality must be equal to the oscillation frequency (as follows from dimensional considerations), so that

$$\gamma_4 \approx \omega \left(\frac{v_\sim}{U}\right)^4. \tag{65}$$

Substituting $v_\sim \approx eE/m\omega$, $U \approx \sqrt{T/m}$, we obtain (64).

The frequency γ_4 is responsible for a number of characteristic features of a turbulent plasma. The decay time for the turbulence spectrum of plasma waves is approximately γ_4^{-1}; the frequency γ_4 (or the mean-free-path for plasma wave collisions) also determines the energy flux q in a gas of plasma waves; $q \approx (v_T^2/\gamma_4)\nabla E^2$; the damping for a nonlinear plasma wave of finite amplitude is also of order γ_4.

As we have noted above, the structure of the collision integral that describes ion waves colliding with ion waves is more complicated than for the plasma waves; the frequency of collisions in a gas of ion-acoustic waves can be estimated from Eq. (65) by making the substitutions

$$v_\sim \approx eE/M\omega; \quad U \approx \sqrt{T/M}; \quad \omega \approx \omega_{0i};$$

$$\gamma_4^s \approx \omega_{0i}\left(\frac{E^2}{nT}\right)^2. \tag{66}$$

It is possible to have situations in which the description of the physical effects in a weakly turbulent plasma requires higher-order processes than the emission and absorption of plasmons by particles or the three- and four-plasmon processes we have considered above. Some of the higher-order processes are scattering of a plasmon by a particle (Fig. 11a) and simultaneous absorption (Fig. 11b), or emission (Fig. 11c), of two plasmons by a particle. The need for considering these processes can arise because the frequency and wave-vector conservation rules do not allow absorption or emission of a single plasmon by a particle [26].

The collision term for plasmons described by the diagram in Fig. 11a is of the form

$$\dot{N}_1 = \sum w_{12;\,34}\{-N_1(N_2+1)f_3(1-f_4)+ \\ +(N_1+1)N_2(1-f_3)f_4\} \tag{67}$$

(the summation is carried out taking account of the conservation of frequency and wave vector); in a rarefied plasma this yields the following expression for the relative change in the number of plasmons:

$$\frac{\dot{N}_1}{N_1} = -\sum w_{12;\,34}N_2(f_3-f_4).$$

Proceeding in similar fashion, we can find the contribution of this process in the particle-collision term.

In treating the dynamics of a turbulent plasma, it may be necessary to sum the series in perturbation theory just as is done in solid-state theory; however, in most investigations of a weakly turbulent plasma, taking account of the emission or absorption of plasmons by electrons and ions and of three- and four-plasmon processes is sufficient.

Thus, the quasi-linear equations (8)-(9) with the plasmon collision integrals (54), (57), (62), and (67), represent a closed system for the investigation of a weakly turbulent plasma.

REFERENCES

1. L. D. Landau, ZhÉTF (J. Exptl. Theoret. Phys. USSR) 16, 25 (1946).
2. A. A. Vlasov, Many-Particle Theory, Gostekhizdat, Moscow, 1950.
3. N. N. Bogolyubov, Dynamical Problems in Statistical Physics, Gostekhizdat, Moscow, 1946.
4. Vedenov, Velikhov, and Sagdeev, Usp. Fiz. Nauk 73, 701 (1961), Soviet Phys. Uspekhi 4, 332 (1962).
5. Propagation of Waves in a Plasma (Russian translation), Problems of

Contemporary Physics, Gostekhizdat, Moscow, 1952.
6. L. D. Landau, ZhÉTF (J. Exptl. Theoret. Phys. USSR) 7, 203 (1937).
7. B. I. Dabydov, Plasma Physics and the Problem of Controlled Thermonuclear Reactions (translated from the Russian), Pergamon Press, New York, 1959, Vol. I.
8. F. E. Low, Proc. Roy. Soc. A248, 282 (1958).
9. Yu. F. Romanov and G. Filippov, ZhÉTF 40, 123 (1961), Soviet Phys. JETP 13, 87 (1961).
10. J. E. Drummond ed., Plasma Physics, McGraw-Hill, New York, 1960.
11. D. Pines and R. Schrieffer, Phys. Rev. 125, 804 (1962).
12. O. V. Konstantinov and V. I. Perel', ZhÉTF 39, 861 (1960), Soviet Phys. JETP 12, 597 (1961).
13. Vedenov, Velikhov, and Sagdeev, Nucl. Fusion 1, 82 (1961).
14. Vedenov, Velikhov, and Sagdeev, Nuclear Fusion, Supplement, 1962, Part 2, p. 465.
15. A. A. Vedenov,Atomnaya énergiya 13, 5 (1962), Atomic Energy 13, 591 (1963).
16. I. F. Kharchenko et. al., 1st International Conference on Plasma Physics and Controlled Thermonuclear Research, Salzburg, 1961.
17. A. A. Vedenov and A. I. Larkin, ZhÉTF 36, 1133 (1959), Soviet Phys. JETP 9, 806 (1959).
18. W. E. Drummond and D. Pines, Nuclear Fusion, Supplement, 1962, Part 3, p. 1049.
19. A. A. Vedenov and E. P. Velikhov, ZhÉTF 43, 963 (1962), Soviet Phys. JETP 16, 682 (1963).
20. A. A. Vedenov, DAN 147, 334 (1962), Soviet Phys. Doklady 7, 100 (1963).
21. L. H. Putnam et al., Phys. Rev. Letters 7, 77 (1961).
22. A. A. Vedenov and E. P. Velikhov, DAN 146, 65 (1962), Soviet Phys. Doklady 7, 801 (1963).
23. E. K. Zavoiskii, Atomnaya énergiya 14, 57 (1963), Atomic Energy 14, 5 (1963).
24. A. B. Mikhailovskii, Reviews of Plasma Physics (translated from the Russian), Consultants Bureau, New York, 1967, Vol. 3.
25. C. Etievant and M. Perulli, Compt. Rend. 255, 855 (1962).
26. K. Matsura and K. Ogawa, Progr. Theoret. Phys. (Japan) 28, 946 (1962).
27. Camac et al., Nuclear Fusion, Supplement, 1962, Part 2, p. 423.
28. A. A. Galeev and V. I. Karpman, ZhÉTF 44, 592 (1963), Soviet Phys. JETP 17, 403 (1963).