advanced, one would rather suspect that the fluctuations of the observables are increased by the loss of information. This is actually true for microscopic variables like the positions and momenta of individual particles. However, if only the so-called macroscopic observables are considered, that is, roughly what was accessible to the more primitive experimental arts of an earlier epoch, then deterministic features arise. Their origin is simply that statistically independent quantities are being averaged: if $a = (1/N)\sum_{j=1}^{N} a_j$, where $w(a_i a_j) = w(a_i)w(a_j)$ for $i \neq j$, then

$$(\Delta a)^2 = \frac{1}{N^2}\Big[w\Big(\sum_{j,k}(a_j a_k)\Big) - \sum_{j,k} w(a_j)w(a_k)\Big] = \frac{1}{N^2}\sum_{j=1}^{N}(\Delta a_j)^2.$$

Thus $\Delta a \sim N^{-1/2}$, and for sufficiently large N the deviations from the average are negligible. We shall learn that in the quantum-theoretical formalism such an a approaches a multiple of the identity operator as $N \to \infty$. The limiting coefficient depends on the representation of the algebra.

Let us verify the phenomena described above in two explicitly soluble models. Of necessity they will lack some of the complications arising in reality, but they exhibit the important features. They are embryonic forms of systems of fermions and bosons.

1.1.1 The Chain of Spins

Let the algebra of observables of the total system be generated by $\boldsymbol{\sigma}_j$, $j = 1, \ldots, N$, where each $\boldsymbol{\sigma}_j$ is a copy of the usual Pauli matrices $\boldsymbol{\sigma}$. Instead of Cartesian components we use $\sigma \equiv \sigma^z$ and $\sigma^{\pm} \equiv (\sigma^x \pm i\sigma^y)/2$, which satisfy the commutation relations

$$\begin{aligned}\left[\sigma_j, \sigma_k^{\pm}\right] &= \pm\delta_{jk} 2\sigma_k^{\pm}, \\ \left[\sigma_j^{+}, \sigma_k^{-}\right] &= \delta_{jk}\sigma_k.\end{aligned} \tag{1.1}$$

The chain is closed by the identification of $\boldsymbol{\sigma}_{j+N}$ with $\boldsymbol{\sigma}_j$, and the Hamiltonian that determines the time-evolution will be assumed to be of the form

$$H = B\sum_{j=1}^{N}\mu_j\sigma_j + \sum_{n=1}^{N-1}\sum_{j=1}^{N}\sigma_j\sigma_{j+n}\varepsilon(n). \tag{1.2}$$

The physical meaning of this is that the spins are coupled with magnetic moments μ_j to an external magnetic field B, and in addition there is an Ising-like spin-spin interaction with the nth neighbor. The strength $\varepsilon(n)$ of this interaction is a function that can be specified later, and the periodicity allows us to assume $\varepsilon(n) = 0$ for $n > N/2$. If the contributions to H are denoted as in

$$H \equiv H_0 + \sum_n H_n, \tag{1.3}$$

then the H_k commute with one another and with the σ_j. They are therefore constant in time, and the time-evolution of σ^+ and $\sigma^- = (\sigma^+)^*$ can be calculated easily from the relationship

$$f(\sigma)\sigma^+ = \sigma^+ f(\sigma + 2), \tag{1.4}$$

which follows from (1.1.2). We find

$$\begin{aligned}\sigma_k^+(t) &= (\sigma_k^-(t))^* = \sigma_k^+(0)\exp\{2it[B\mu_k + \sum_n \varepsilon(n)(\sigma_{k+n} + \sigma_{k-n})]\}\\ &= \sigma_k^+(0)\exp(2itB\mu_k)\prod_n(\cos 2t\varepsilon(n) + i\sigma_{k+n}\sin 2t\varepsilon(n))(\cos 2t\varepsilon(n)\\ &\quad + i\sigma_{k-n}\sin 2t\varepsilon(n)),\end{aligned} \tag{1.5}$$

where $a(t) = \exp(iHt)a\exp(-iHt)$.

The time-evolution consists of Larmor precession in the external field and a kind of diffusion along the chain due to the spin-spin interaction. Suppose that the state at $t = 0$ is pure and has the form of a product, where the spins have a 3-component s and σ_k^+ has phase α_k:

$$\langle\sigma_k(0)\rangle = s, \quad \langle\sigma_k^+(0)\rangle = \frac{1}{2}\sqrt{1-s^2}\exp(i\alpha_k), \quad \langle\prod_j \boldsymbol{\sigma}_j\rangle = \prod_j\langle\boldsymbol{\sigma}_j\rangle. \tag{1.6}$$

Then

$$\begin{aligned}\langle\sigma_k^+(t)\rangle &= \frac{1}{2}\sqrt{1-s^2}\exp\{i(\alpha_k + 2tB\mu_k)\}f^2(t),\\ f(t) &= \prod_{n=1}^{N/2}(\cos 2t\varepsilon(n) + is\sin 2t\varepsilon(n)).\end{aligned} \tag{1.7}$$

If N is finite, then f is almost periodic, and if $N = \infty$, then $f(t)$ will generally tend to zero as $t \to \infty$ (supposing that $\varepsilon(n)$ tends to zero in such a way that the infinite product makes sense). To make this more explicit, let us consider the special case $s = 0$ and $\varepsilon(n) = 2^{-n-1}$. If $N = \infty$, then f satisfies the equation

$$f(t) = \prod_{n=1}^{\infty}\cos 2^{-n}t = \frac{f(2t)}{\cos t}. \tag{1.8}$$

Since f is an entire function, this functional equation and the condition $f(0) = 1$ determine f uniquely – differentiate (1.8) to get the Taylor series of f. Since the function $(\sin t)/t$ satisfies (1.8), it equals f. Hence, as $N \to \infty$, the expectation value of $\sigma^\pm$ approaches zero. For finite N it follows from (1.8) that

$$f_N(t) = \prod_{n=1}^{N/2}\cos 2^{-n}t = \frac{\sin t}{t}\left[\frac{\sin t2^{-N/2}}{t2^{-N/2}}\right]^{-1}. \tag{1.9}$$

Therefore, as discussed earlier, the recurrence time $2^{N/2}/\pi$ grows exponentially with N, while the time it takes to reach equilibrium is independent of N.

To summarize, we have ascertained that for $N \to \infty$ the initially pure state of the algebra reduced to one spin tends as $t \to \infty$ to $\langle\sigma\rangle = s$, $\langle\sigma^\pm\rangle = 0$, which corresponds to a mixture:

$$\langle \boldsymbol{\sigma} \rangle = \mathrm{Tr}(\rho \boldsymbol{\sigma}), \quad \rho = \frac{\exp(-\eta\sigma)}{\mathrm{Tr}\exp(-\eta\sigma)}, \quad \tanh\eta = s. \tag{1.10}$$

Even though the expectation values of the $\sigma_k^{\pm}$ go to zero, their fluctuations remain nonzero, since $\sigma_k^{+}\sigma_k^{-} = (1+\sigma_k)/2$ is constant. The average magnetization

$$\mathbf{M}_N(t) = \frac{1}{N}\sum_k \boldsymbol{\sigma}_k(t) \tag{1.11}$$

works differently. In the state (1.6) of our example, $\langle M_N^z \rangle = s$, whereas $\langle M_N^{\pm} \rangle$ is $O(N^{-1/2})$, provided either that the initial phases are disordered or that the $\sigma_k^{\pm}$ get out of phase after a while because the μ_k differ. The latter situation can in fact be undone by a sudden reversal of B, in the spin-echo effect. If $N = \infty$, the diffusion caused by suitable $\varepsilon(n)$ is irreversible, and $lim_{t\to\infty}\langle M_\infty^{\pm}(t)\rangle = 0$. At $t = 0$ the fluctuations are $O(N^{-1/2})$ and remain at this magnitude for all time: If $\sigma_k^{+}(t)\sigma_{k'}^{-}(t)$ is calculated by multiplying together two expressions of the form (1.1.6), then it should be recalled that $\sigma^2 = 1$. However, if the function $\varepsilon(n)$ falls off sufficiently rapidly with n, then the σ^2 terms make little difference for large $k-k'$, and the argument given earlier for the deviations of statistically independent quantities remains valid.

1.1.2 Chain of Oscillators

Now represent the total system by positions and momenta $q_1, \ldots, q_N, p_1, \ldots, p_N$, such that $[q_j, p_k] = i\delta_{jk}$, and let the time-evolution be determined by

$$H = \sum_{j=1}^{N} \frac{1}{2}\left(p_j^2 + (q_j - q_{j+1})^2\right). \tag{1.12}$$

This Hamiltonian contains interactions only between nearest neighbors, and the chain can be closed by the condition of periodicity $q_{j+N} = q_j$, $p_{j+N} = p_j$. The masses and force constants have been set to 1, which amounts to measuring the time in units of the natural period of oscillation. The equations of motion are

$$j = p_j, \quad \dot{p}_j = q_{j+1} + q_{j-1} - 2q_j. \tag{1.13}$$

With a periodic extension of the variables, $\xi_1, \ldots, \xi_{2N}$, such that

$$\xi_{2n} = p_n, \quad \xi_{2n+1} = q_{n+1} - q_n, \tag{1.14}$$

they are put into the form

$$\dot{\xi}_j = \xi_{j+1} - \xi_{j-1}. \tag{1.15}$$

The variables ξ_n satisfy

$$\xi_{n+2N} = \xi_n, \quad \sum_n \xi_{2n+1} = 0.$$